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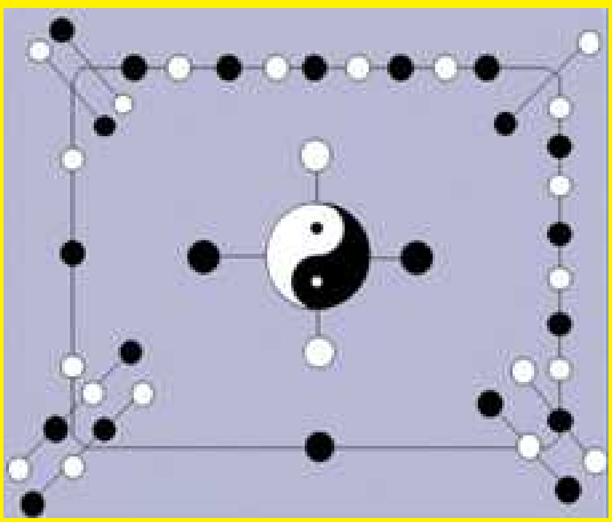
Edited By Linfan MAO



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Mathematical Combinatorics

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The Madis of Chinese Academy of Sciences

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Lucas Graceful Labeling for Some Graphs

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Abstract: A *Smarandache-Fibonacci triple* is a sequence $S(n)$, $n \geq 0$ such that $S(n) = S(n-1) + S(n-2)$, where $S(n)$ is the Smarandache function for integers $n \geq 0$. Clearly, it is a generalization of *Fibonacci sequence* and *Lucas sequence*. Let G be a (p,q)-graph and $\{S(n)|n \geq 0\}$ a Smarandache-Fibonacci triple. An bijection $f\colon V(G) \to \{S(0), S(1), S(2), \ldots, S(q)\}$ is said to be a *super Smarandache-Fibonacci graceful graph* if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{S(1), S(2), \ldots, S(q)\}$. Particularly, if $S(n), n \geq 0$ is just the Lucas sequence, such a labeling $f: V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}$ $(a \epsilon N)$ is said to be *Lucas graceful labeling* if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection on to the set $\{l_1, l_2, \cdots, l_q\}$. Then G is called *Lucas graceful graph* if it admits Lucas graceful labeling. Also an injective function $f: V(G) \to \{l_0, l_1, l_2, \cdots, l_q\}$ is said to be strong Lucas graceful labeling if the induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, ..., l_q\}$. G is called strong Lucas graceful graph if it admits strong Lucas graceful labeling. In this paper, we show that some graphs namely P_n, $P_n^+ - e$, $S_{m,n}$, $F_m@P_n$, $C_m@P_n$, $K_{1,n} \odot 2P_m$, $C_3@2P_n$ and $C_n@K_{1,2}$ admit Lucas graceful labeling and some graphs namely $K_{1,n}$ and F_n admit strong Lucas graceful labeling.

Key Words: Smarandache-Fibonacci triple, super Smarandache-Fibonacci graceful graph, Lucas graceful labeling, strong Lucas graceful labeling.

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§1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A path of length n is denoted by P_n. A cycle of length n is denoted by C_n.G^+ is a graph obtained from the graph G by attaching a pendant vertex to each vertex of G. The concept of graceful labeling was introduced by Rosa [3] in 1967.

A function f is a graceful labeling of a graph G with q edges if f is an injection from

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the vertices of G to the set $\{1, 2, 3, \cdots, q\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. The notion of Fibonacci graceful labeling was introduced by K.M.Kathiresan and S.Amutha [4]. We call a function, a Fibonacci graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, ..., F_q\}$, where F_q is the q^{th} Fibonacci number of the Fibonacci series $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, ...$, and each edge uv is assigned the label $|f(u) - f(v)|$. Based on the above concepts we define the following.

Let G be a (p, q) -graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}$, $(a \in N)$, is said to be Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11, \cdots,$. Then G is called Lucas graceful graph if it admits Lucas graceful labeling. Also an injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_q\}$ is said to be strong Lucas graceful labeling if the induced edge labeling $f_1(uv) = |f(u)-f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$. Then G is called strong Lucas graceful graph if it admits strong Lucas graceful labeling. In this paper, we show that some graphs namely P_n, $P_n^+ - e$, $S_{m,n}$, $F_m@P_n$, $C_m@P_n$, $K_{1,n} \odot 2P_m$, $C_3@2P_n$ and $C_n@K_{1,2}$ admit Lucas graceful labeling and some graphs namely $K_{1,n}$ and F_n admit strong Lucas graceful labeling. Generally, let $S(n)$, $n \geq 0$ with $S(n) = S(n-1) + S(n-2)$ be a *Smarandache-Fibonacci triple*, where $S(n)$ is the Smarandache function for integers $n \geq 0$. An bijection $f : V(G) \to \{S(0), S(1), S(2), \ldots, S(q)\}$ is said to be a *super Smarandache-Fibonacci graceful graph* if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{S(1), S(2), \cdots, S(q)\}$.

§2. Lucas graceful graphs

In this section, we show that some well known graphs are Lucas graceful graphs.

Definition 2.1 *Let G be a (p, q) -graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a, \}$, $(a \in N)$ is said to be Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{l_1, l_2, \cdots, l_q\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 = 7, l_5 = 11, \cdots,$. Then G is called Lucas graceful graph if it admits Lucas graceful labeling.*

Theorem 2.2 *The path P_n is a Lucas graceful graph.*

Proof Let P_n be a path of length n having $(n+1)$ vertices namely $v_1, v_2, v_3, \cdots, v_n, v_{n+1}$. Now, $|V(P_n)| = n + 1$ and $|E(P_n)| = n$. Define $f : V(P_n) \to \{l_0, l_1, l_2, \cdots, l_a, \}, a \in N$ by $f(u_i) = l_{i+1}, 1 \leq i \leq n$. Next, we claim that the edge labels are distinct. Let

$$\begin{aligned} E &= \{f_1(v_i v_{i+1}) : 1 \leq i \leq n\} = \{|f(v_i) - f(v_{i+1})| : 1 \leq i \leq n\} \\ &= \{|f(v_1) - f(v_2)|, |f(v_2) - f(v_3)|, \cdots, |f(v_n) - f(v_{n+1})|, \} \\ &= \{|l_2 - l_3|, |l_3 - l_4|, \cdots, |l_{n+1} - l_{n+2}|\} = \{l_1, l_2, \cdots, l_n\}. \end{aligned}$$

So, the edges of P_n receive the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, the path P_n is a Lucas graceful graph. □

Example 2.3 *The graph P_6 admits Lucas graceful Labeling, such as those shown in Fig.*1 *following.*

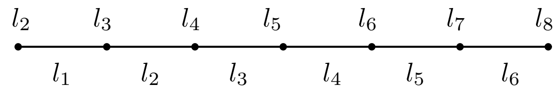

Fig.1

Theorem 2.4 *$P_n^+ - e, (n \geq 3)$ is a Lucas graceful graph.*

Proof Let $G = P_n^+ - e$ with $V(G) = \{u_1, u_2, \cdots, u_{n+1}\} \bigcup \{v_2, v_3, \cdots, v_{n+1}\}$ be the vertex set of G. So, $|V(G)| = 2n+1$ and $|E(G)| = 2n$. Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a, \}, a \in N$, by

$$f(u_i) = l_{2i-1}, 1 \leq i \leq n+1 \text{ and } f(v_j) = l_{2(j-1)}, 2 \leq j \leq n+1.$$

We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \{f_1(u_i u_{i+1}) : 1 \leq i \leq n\} = \{|f(u_i) - f(u_{i+1})| : 1 \leq i \leq n\} \\ &= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, \cdots, |f(u_n) - f(u_{n+1})|\} \\ &= \{|l_1 - l_3|, |l_3 - l_5|, \cdots, |l_{2n-1} - l_{2n+1}|\} = \{l_2, l_4, \cdots, l_{2n}\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \{f_1(u_i v_j) : 2 \leq i,\ j \leq n\} \\ &= \{|f(u_2) - f(v_2)|, |f(u_3) - f(v_3)|, \cdots, |f(u_{n+1}) - f(v_{n+1})|\} \\ &= \{|l_3 - l_2|, |l_5 - l_4|, \cdots, |l_{2n+1} - l_{2n}|\} = \{l_1, l_3, \cdots, l_{2n-1}\}. \end{aligned}$$

Now, $E = E_1 \cup E_2 = \{l_1, l_3, \cdots, l_{2n-1}, l_{2n}\}$. So, the edges of G receive the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, $P_n^+ - e, (n \geq 3)$ is a Lucas graceful graph. $\square$

Example 2.5 The graph $P_8^+ - e$ admits Lucas graceful labeling, such as thsoe shown in Fig.2.

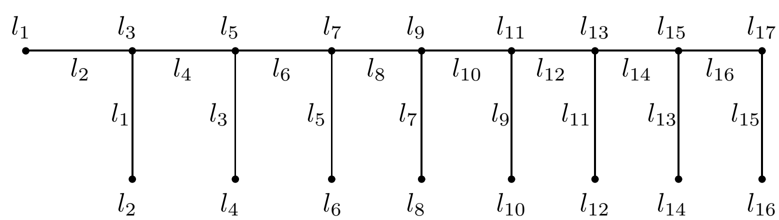

Fig.2

Definition 2.6([2]) *Denote by $S_{m,n}$ such a star with n spokes in which each spoke is a path of length m.*

Theorem 2.7 *The graph $S_{m,n}$ is a Lucas graceful graph when m is odd and $n \equiv 1, 2 (mod\ 3)$.*

Proof Let $G = S_{m,n}$ and let $V(G) = \left\{u_j^i : 1 \leq i \leq m \text{ } and \text{ } 1 \leq j \leq n\right\}$ be the vertex set of $S_{m,n}$. Then $|V(G)| = mn + 1$ and $|E(G)| = mn$. Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a,\}, a \in N$ by

$f(u_0) = l_0$ for $i = 1, 2, \cdots, m-2$ and $i \equiv 1(mod\ 2)$;
$f\left(u_j^i\right) = l_{n(i-1)+2j-1}, 1 \leq j \leq n$ for $i = 1, 2, \cdots, m-1$ and $i \equiv 0(mod\ 2)$;
$f\left(u_j^i\right) = l_{ni+2-2j}, 1 \leq j \leq n$ and for $s = 1, 2, \cdots, \dfrac{n}{3}$,
$f\left(u_j^m\right) = l_{n(m-1)+2(j+1)-3s}, 3s-2 \leq j \leq 3s$.

We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \left\{f_1\left(u_0\ u_1^i\right)\right\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \left\{\left|f\left(u_0\right) - f\left(u_1^i\right)\right|\right\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \left\{\left|l_0 - l_{n(i-1)+1}\right|\right\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m} \left\{l_{n(i-1)+1}\right\} \\ &= \left\{l_1, l_{2n+1}, l_{4n+1}, \cdots, l_{n(m-1)+1}\right\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-1} \left\{f_1\left(u_0\ u_1^i\right)\right\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-1} \left\{\left|f\left(u_0\right) - f\left(u_1^i\right)\right|\right\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-1} \left\{\left|l_0 - l_{ni}\right|\right\} = \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-1} \left\{l_{ni}\right\} = \left\{l_{2n}, l_{4n}, \cdots, l_{n(m-1)}\right\} \end{aligned}$$

$$\begin{aligned} E_3 &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{f_1\left(u_j^i u_{j+1}^i\right) : 1 \leq j \leq n-1\right\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{\left|f\left(u_j^i\right) - f\left(u_{j+1}^i\right)\right| : 1 \leq j \leq n-1\right\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{\left|l_{n(i-1)+2j-1} - l_{n(i-1)+2j+1}\right| : 1 \leq j \leq n-1\right\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{l_{n(i-1)+2j} : 1 \leq j \leq n-1\right\} \\ &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{l_{n(i-1)+2}, l_{n(i-1)+4}, \cdots, : l_{n(i-1)+2(n-1)}\right\} \\ &= \left\{l_2, l_{2n+2}, ..., l_{n(m-3)+2}\right\} \cup \left\{l_4, l_{2n+4}, \cdots, l_{n(m-3)+4}\right\} \cup \cdots \\ &\quad \cup \left\{l_{2n-2}, l_{4n-2}, ..., l_{n(m-3)+2n-2}\right\}, \end{aligned}$$

$$
\begin{aligned}
E_4 &= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{f_1\left(u_j^i\ u_{j+1}^i\right) : 1 \le j \le n-1\right\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{\left|f\left(u_j^i\right) - f\left(u_{j+1}^i\right)\right| : 1 \le j \le n-1\right\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{\left|l_{ni-2j+2} - l_{ni-2j}\right| : 1 \le j \le n-1\right\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{l_{ni-2j+1} : 1 \le j \le n-1\right\} \\
&= \bigcup_{\substack{i=1 \\ i\equiv 1(mod\ 2)}}^{m-2} \left\{l_{ni-1}, l_{ni-3}, \cdots, l_{ni-(2n-3)}\right\} \\
&= \left\{l_{2n-1}, l_{2n-3}, \cdots, l_3, l_{4n-1}, l_{4n-3}, \cdots, l_{2n+3}, l_{n(m-1)-1}, \cdots, l_{n(m-1)-(2n-3)}\right\}.
\end{aligned}
$$

For $n \equiv 1(mod\ 3)$, let

$$
\begin{aligned}
E_5 &= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{f_1\left(u_j^m\ u_{j+1}^m\right) : 3s-2 \le j \le 3s-1\right\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{\left|f\left(u_j^m\right) - f\left(u_{j+1}^m\right)\right| : 3s-2 \le j \le 3s-1\right\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{\left|l_{n(m-1)+2j-3s+2} - l_{n(m-1)+2j-3s+4}\right| : 3s-2 \le j \le 3s-1\right\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{l_{n(m-1)+2j-3s+2} : 3s-2 \le j \le 3s-1\right\} = \bigcup_{s=1}^{\frac{n-1}{3}} \left\{l_{n(m-1)+3s-1}, l_{n(m-1)+3s+1}\right\} \\
&= \left\{l_{n(m-1)+2}, l_{n(m-1)+4}, l_{n(m-1)+5}, l_{n(m-1)+7}, \cdots, l_{n\ m-2}, l_{mn}\right\}.
\end{aligned}
$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, \cdots, \dfrac{n-1}{3}$. Let

$$
\begin{aligned}
E_6 &= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{\left|f_1\left(u_{3s}^m\ u_{3s+1}^m\right)\right|\right\} = \bigcup_{s=1}^{\frac{n-1}{3}} \left\{\left|f\left(u_{3s}^m\right) - f\left(u_{3s+1}^m\right)\right|\right\} \\
&= \left\{\left|f\left(u_3^m\right) - f\left(u_4^m\right)\right|, \left|f\left(u_6^m\right) - f\left(u_7^m\right)\right|, \left|f\left(u_9^m\right) - f\left(u_{10}^m\right)\right|, \cdots, \left|f\left(u_{n-1}^m\right) - f\left(u_n^m\right)\right|\ \right\} \\
&= \left\{\left|l_{n(m-1)+5} - l_{n(m-1)+4}\right|, \left|l_{n(m-1)+8} - l_{n(m-1)+7}\right|, \cdots, \left|l_{n(m-1)+n+1} - l_{n(m-1)+n}\right|\right\} \\
&= \left\{l_{n(m-1)+3}, l_{n(m-1)+6}, \cdots, l_{n(m-1)+n-1}\right\} = \left\{l_{n(m-1)+3}, l_{n(m-1)+6}, \cdots, l_{nm-1}\right\}.
\end{aligned}
$$

For $n \equiv 2(mod\ 3)$, let

$$\begin{aligned}
E_5' &= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ f_1\left(u_j^m\ u_{j+1}^m\right) : 3s-2 \le j \le 3s-1 \right\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ \left| f\left(u_j^m\right) - f\left(u_{j+1}^m\right) \right| : 3s-2 \le j \le 3s-1 \right\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ \left| l_{n(m-1)+2j-3s+2} - l_{n(m-1)+2j-3s+4} \right| : 3s-2 \le j \le 3s-1 \right\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ l_{n(m-1)+2j-3s+3} : 3s-2 \le j \le 3s-1 \right\} = \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ l_{n(m-1)+3s-1}, l_{n(m-1)+3s+1} \right\} \\
&= \left\{ l_{n(m-1)+2}, l_{n(m-1)+4}, l_{n(m-1)+5}, l_{n(m-1)+7}, \cdots, l_{n(m-1)+n-2}, l_{n(m-1)+n} \right\}.
\end{aligned}$$

We determine the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, 3, ..., \dfrac{n-1}{3}$.

$$\begin{aligned}
Let\ E_6' &= \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ f_1\left(u_{3s}^m\ u_{3s+1}^m\right) \right\} = \bigcup_{s=1}^{\frac{n-1}{3}} \left\{ \left| f\left(u_{3s}^m\right) - f\left({}_{3s+1}^{m}\right) \right| \right\} \\
&= \left\{ \left| f\left(u_3^m\right) - f\left(u_4^m\right) \right|, \left| f\left(u_6^m\right) - f\left(u_7^m\right) \right|, \left| f\left(u_9^m\right) - f\left(u_{10}^m\right) \right|, \cdots, \left| f\left(u_{n-1}^m\right) - f\left(u_n^m\right) \right| \right\} \\
&= \left\{ \left| l_{n(m-1)+5} - l_{n(m-1)+4} \right|, \left| l_{n(m-1)+8} - l_{n(m-1)+7} \right|, \cdots, \left| l_{n(m-1)+n+1} - l_{n(m-1)+n} \right| \right\} \\
&= \left\{ l_{n(m-1)+3}, l_{n(m-1)+6}, \cdots, l_{nm-1} \right\}.
\end{aligned}$$

Now, $E = \bigcup_{i=1}^{6} E_i$ if $n \equiv 1(mod\ 3)$ and $E = \left(\bigcup_{i=1}^{6} E_i\right) \bigcup E_5' \bigcup E_6'$ if $n \equiv 2(mod\ 3)$. So the edges of $S_{m,n}$ (when m is odd and $n \equiv 1, 2(mod\ 3)$), receive the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, $S_{m,n}$ is a Lucas graceful graph if m is odd, $n \equiv 1, 2(mod\ 3)$. $\square$

Example 2.8 The graphs $S_{5,4}$ and $S_{5,5}$ admit Lucas graceful labeling, such as those shown in Fig.3 and Fig 4.

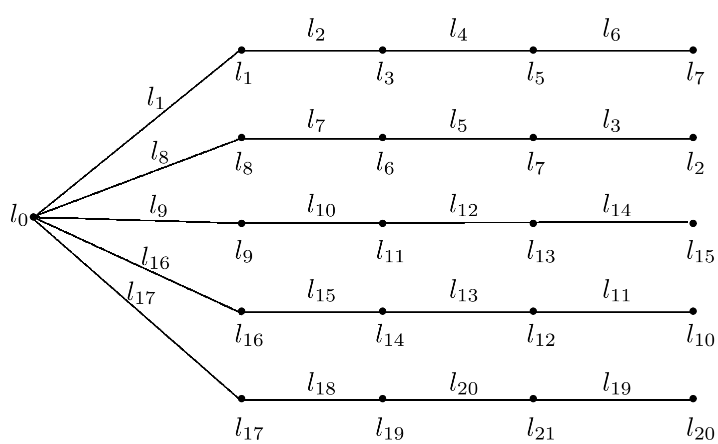

Fig.3

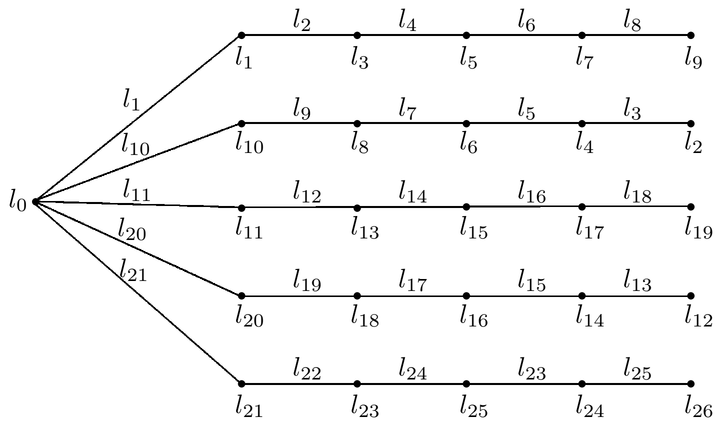

Fig.4

Definition 2.9([2]) *The graph $G = F_m@P_n$ consists of a fan F_m and a path P_n of length n which is attached with the maximum degree of the vertex of F_m.*

Theorem 2.10 *$F_m@P_n$ is a Lucas graceful labeling when $n \equiv 1, 2$ $(mod\ 3)$.*

Proof Let $v_1, v_2, ..., v_m, v_{m+1}$ and u_0 be the vertices of a fan F_m and $u_1, u_2, \cdots, u_n$ be the vertices of a path P_n. Let $G = F_m@P_n$. Then $|V(G)| = m + n + 2$ and $|E(G)| = 2m + n + 1$. Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a,\}, a \ \epsilon \ N$, by $f(u_0) = l_0,\ f(v_i) = l_{2i-1}, 1 \le i \le m + 1$. For $s = 1, 2, \cdots, \dfrac{n-1}{3}$ *or* $\dfrac{n-2}{3}$ according as $n \equiv 1(mod\ 3)$ or $n \equiv 2(mod\ 3)$, $f(u_j) = l_{2m+2j-3s+3}, 3s - 2 \le j \le 3s$.

We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \{f_1(v_i\ v_{i+1}) : 1 \le i \le m\} = \{|f(v_i) - f\,(v_{i+1})| : 1 \le i \le m\} \\ &= \{|l_{2i-1} - l_{2i+1}| : 1 \le i \le m\} \\ &= \{l_{2i} : 1 \le i \le m\} = \{l_2, l_4, \cdots, l_{2m}\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \{f_1\,(u_0 v_i) : 1 \le i \le m+1\} = \{|f(u_0) - f(v_i)|\, 1 \le i \le m+1\} \\ &= \{|l_0 - l_{2i-1}| : 1 \le i \le m+1\} \\ &= \{l_{2i-1} : 1 \le i \le m+1\} = \{l_1, l_3, \cdots, l_{2m+1}\} \end{aligned}$$

and

$$E_3 = \{f_1\,(u_0 u_1)\} = \{|f(u_0) - f(u_1)|\} = \{|l_0 - l_{2m+2}|\} = \{l_{2m+2}\}$$

For $s = 1, 2, 3, \cdots, \dfrac{n-1}{3}$ and $n \equiv 1(mod\ 3)$, let

$$\begin{aligned}
E_4 &= \bigcup_{s=1}^{\frac{n-1}{3}} \{f_1(u_j, u_{j+1}) : 3s-2 \le j \le 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \{|f(u_j) - f(u_{j+1})| : 3s-2 \le j \le 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \{|l_{2m+2j+3-3s} - l_{2m+2j+5-3s}| : 3s-2 \le j \le 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} (l_{2m+2j+4-3s} : 3s-2 \le j \le 3s-1) \\
&= \{l_{2m+2j-2} : 4 \le j \le 5\} \bigcup \{l_{2m+2j-5} : 7 \le j \le 8\} \bigcup \cdots \\
&\quad \bigcup \{l_{2m+2j-n+4} : n-3 \le j \le n-2\} \\
&= \{l_{2m+6}, l_{2m+8}\} \cup \{l_{2m+9}, l_{2m+11}\} \bigcup \cdots \bigcup \{l_{2m+n-2}, l_{2m+n}\} \\
&= \{l_{2m+6}, l_{2m+8}, l_{2m+9}, l_{2m+11}, \cdots, l_{2m+n-2}, l_{2m+n}\}
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, 3, \cdots, \dfrac{n-1}{3}$, $n \equiv 1(mod\ 3)$. Let

$$\begin{aligned}
E_5 &= \bigcup_{s=1}^{\frac{n-1}{3}} \{f_1(u_j\ u_{j+1}) : j = 3s\} = \bigcup_{s=1}^{\frac{n-1}{3}} \{|f(u_j) - f(u_{j+1})| : j = 3s\} \\
&= \bigcup_{s=1}^{\frac{n-1}{3}} \{|l_{2m+2j+3-3s} - l_{2m+2j+5-3s}| : j = 3s\} \\
&= \{|l_{2m+2j} - l_{2m+2j-1}| : j = 3\} \cup \{|l_{2m+2j-3} - l_{2m+2j-4}| : j = 6\} \bigcup \cdots \\
&\quad \bigcup \{|l_{2m+2j} - l_{2m+2j-1}| : j = n-1\} \\
&= \{l_{2m+2j-2} : j = 3\} \cup \{l_{2m+2j-5} : j = 6\} \cup, \cdots, \cup \{l_{2m+2j-n+3} : j = n-1\} \\
&= \{l_{2m+4}, l_{2m+7}, \cdots, l_{2m+n+1}\}.
\end{aligned}$$

For $s = 1, 2, 3, \cdots, \dfrac{n-2}{3}$ and $n \equiv 2(mod\ 3)$, let

$$\begin{aligned}
E_4^{'} &= \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_j\ u_{j+1}) : 3s-2 \le j \le 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_j) - f(u_{j+1})| : 3s-2 \le j \le 3s-1\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{|l_{2m+2j+3-3s} - l_{2m+2j+5-3s}| : 3s-2 \le j \le 3s-1\}
\end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{s=1}^{\frac{n-2}{3}} (l_{2m+2j+4-3s} : 3s-2 \le j \le 3s-1) \\
&= \{l_{2m+2j-2} : 4 \le j \le 5\} \bigcup \{l_{2m+2j-5} : 7 \le j \le 8\} \bigcup \cdots \\
&\quad \bigcup \{l_{2m+2j-n+4} : n-3 \le j \le n-2\} \\
&= \{l_{2m+6}, l_{2m+8}\} \bigcup \{l_{2m+9}, l_{2m+11}\} \bigcup \cdots \bigcup \{l_{2m+n-2}, l_{2m+n}\} \\
&= \{l_{2m+6}, l_{2m+8,2m+9}, l_{2m+11}, \cdots, l_{2m+n-2}, l_{2m+n}\}
\end{aligned}
$$

We determine the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, 3, ..., \frac{n-2}{3}$, $n \equiv 2(mod\ 3)$. Let

$$
\begin{aligned}
E_5^{'} &= \bigcup_{s=1}^{\frac{n-2}{3}} \{f_1(u_j, u_{j+1}) : j = 3s\} \\
&= \bigcup_{s=1}^{\frac{n-2}{3}} \{|f(u_j) - f(u_{j+1})| : j = 3s\} = \bigcup_{s=1}^{\frac{n-2}{3}} \{|l_{2m+2j+3-3s} - l_{2m+2j+5-3s}| : j = 3s\} \\
&= \{|l_{2m+2j} - l_{2m+2j-1}| : j = 3\} \bigcup \{|l_{2m+2j-3} - l_{2m+2j-4}| : j = 6\} \bigcup \cdots \\
&\quad \bigcup \{|l_{2m+2j-n+4} - l_{2m+2j-n+5}| : j = n-1\} \\
&= \{l_{2m+2j-2} : j = 3\} \cup \{l_{2m+2j-5} : j = 6\} \bigcup \cdots \bigcup \{l_{2m+2j-(n-3)} : j = n-1\} \\
&= \{l_{2m+4}, l_{2m+7}, ..., l_{2m+n+1}\}.
\end{aligned}
$$

Now, $E = \bigcup\limits_{i=1}^{5} E_i$ if $n \equiv 1(mod\ 3)$ and $E = \left(\bigcup\limits_{i=1}^{5} E_i\right) \bigcup E_4^{'} \bigcup E_5^{'}$ if $n \equiv 2(mod\ 3)$. So, the edges of $F_m@P_n$ (when$n \equiv 1, 2(mod\ 3)$) are the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, $G = F_m@P_n$ (if $n \equiv 1, 2(mod\ 3)$) is a Lucas graceful labeling. □

Example 2.11 The graph $F_5@P_4$ admits a Lucas graceful labeling shown in Fig.5.

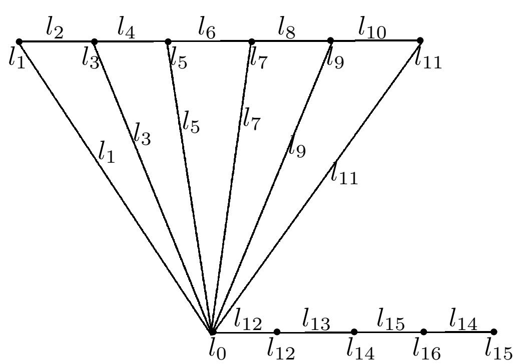

Fig.5

Definition 2.12 ([2]) *The Graph $G = C_m@P_n$ consists of a cycle C_m and a path of P_n of length n which is attached with any one vertex of C_m.*

Theorem 2.13 *The graph $C_m@P_n$ is a Lucas graceful graph when $m \equiv 0(mod\ 3)$ and $n = 1, 2(mod\ 3)$.*

Proof Let $G = C_m@P_n$ and let $u_1, u_2, \cdots, u_m$ be the vertices of a cycle C_m and $v_1, v_2, \cdots, v_n, v_{n+1}$ be the vertices of a path P_n which is attached with the vertex $(u_1 = v_1)$ of C_m. Let $V(G) = \{u_1 = v_1\} \cup \{u_2, u_3, \cdots, u_m\} \cup \{v_2, v_3, ..., v_n, v_{n+1}\}$ be the vertex set of G. So, $|V(G)| = m + n$ and $|E(G)| = m + n$. Define $f : V(G) \to \{l_0, l_1, \cdots, l_a\},\ a \epsilon N$ by $f(u_1) = f(v_1) = l_0$; $f(u_i) = l_{2i-3s}, 3s - 1 \leq j \leq 3s + 1$ for $s = 1, 2, 3, \cdots, \dfrac{m}{3}$, $i = 2, 3, \cdots, m$; $f(v_j) = l_{m+2j-3r}, 3r - 1 \leq j \leq 3r + 1$ for $r = 1, 2, \cdots, \dfrac{n+1}{3}$ and $j = 2, 3, \cdots, n + 1$.

We claim that the edge labels are distinct. Let

$$E_1 = \{f_1(u_1\ u_2)\} = \{|f(u_1) - f(u_2)|\} = (|l_0 - l_1|) = \{l_1\},$$

$$\begin{aligned} E_2 &= \bigcup_{s=1}^{\frac{m}{3}} \{f_1(u_i\ u_{i+1}) : 3s - 1 \leq i \leq 3s \textit{ and } u_{m+1} = u_1\} \\ &= \bigcup_{s=1}^{\frac{m}{3}} \{f_1(u_i) - f(u_{i+1}) : 3s - 1 \leq i \leq 3s \textit{ and } u_{m+1} = u_1\} \\ &= \{|f(u_2) - f(u_3)|, |f(u_3) - f(u_4)|, ..., |f(u_m) - f(u_{m+1})|\} \\ &= \{|l_1 - l_3|, |l_3 - l_5|, |l_4 - l_6|, |l_6 - l_8|, \cdots, |l_m - l_0|\} \\ &= \{l_2, l_4, l_5, l_7, \cdots, l_m\} \end{aligned}$$

We determine the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $s = 1, 2, ..., \dfrac{m}{3} - 1$. Let

$$\begin{aligned} E_3 &= \bigcup_{s=1}^{\frac{m}{3}-1} \{f_1(u_{3s+1}\ u_{3s+2})\} = \bigcup_{s=1}^{\frac{m}{3}-1} \{|f(u_{3s+1}) - f(u_{3s+2})|\} \\ &= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, \cdots, |f(u_{m-2}) - f(u_{m-1})|\} \\ &= \{|l_5 - l_4|, |l_8 - l_7|, \cdots, |l_{m-1} - l_{m-2}|\} \\ &= \{l_3, l_6, \cdots, l_{m-3}\}, \end{aligned}$$

$$\begin{aligned} E_4 &= \{f_1(v_1\ v_2)\} = \{|f(v_1) - f(v_2)|\} = \{|l_0 - l_{m+4-3}|\} \{|l_0 - l_{m+4-3}|\} \\ &= \{|l_0 - l_{m+1}|\} = \{|l_0 - l_{m+1}|\} = \{l_{m+1}\}. \end{aligned}$$

For $n \equiv 1(mod\ 3)$, let

$$\begin{aligned} E_5 &= \bigcup_{r=1}^{\frac{n-1}{3}} \{f_1(v_j\ v_{j+1}) : 3r - 1 \leq j \leq 3r\} \\ &= \bigcup_{r=1}^{\frac{n-1}{3}} \{|f(v_j) - f(v_{j+1})| : 3r - 1 \leq j \leq 3r\} \end{aligned}$$

$$\begin{aligned}
&= \{|f(v_2)-f(v_3)|, |f(v_3)-f(v_4)|, \cdots, |f(v_{n-1})-f(v_n)|\} \\
&= \{|l_{m+4-3}-l_{m+6-3}|, |l_{m+6-3}-l_{m+8-3}|, |l_{m+10-6}-l_{m+12-6}|, |l_{m+12-6}-l_{m+14-6}|, \\
&\quad \cdots, |l_{m+2n-2-n+1}-l_{m+2n-n+1}|\} \\
&= \{|l_{m+1}-l_{m+3}|, |l_{m+3}-l_{m+5}|, |l_{m+4}-l_{m+6}|, |l_{m+6}-l_{m+8}|, \cdots, |l_{m+n-1}-l_{m+n+1}|\} \\
&= \{l_{m+2}, l_{m+4}, l_{m+5}, l_{m+7}, \cdots, l_{m+n}\}.
\end{aligned}$$

We calculate the edge labeling between the end vertex of r^{th} loop and the starting vertex of $(r+1)^{th}$ loop and $r = 1, 2, \cdots, \dfrac{n-1}{3}$. Let

$$\begin{aligned}
E_6 &= \bigcup_{r=1}^{\frac{n-1}{3}} \{f_1(v_{3r+1}\ v_{3r+2})\} = \bigcup_{r=1}^{\frac{n-1}{3}} \{|f(v_{3r+1})-f(v_{3r+2})|\} \\
&= \{|f(v_4)-f(v_5)|, |f(v_7)-f(v_8)|, \cdots, |f(v_{n-2})-f(v_{n-1})|\} \\
&= \{|l_{m+8-3}-l_{m+10-6}|, |l_{m+14-6}-l_{m+16-9}|, \cdots, |l_{m+2n-4-n+2}-l_{m+2n-2-n+1}|\} \\
&= \{|l_{m+5}-l_{m+4}|, |l_{m+8}-l_{m+7}|, \cdots, |l_{m+n-2}-l_{m+n}|\} \\
&= \{l_{m+3}, l_{m+6}, l_{m+9}, \cdots, l_{m+n-1}\}
\end{aligned}$$

For $n \equiv 2(mod\ 3)$, let

$$\begin{aligned}
E_5' &= \bigcup_{r=1}^{\frac{n-1}{3}} \{f_1(v_j\ v_{j+1}) : 3r-1 \le j \le 3r\} = \bigcup_{r=1}^{\frac{n-1}{3}} \{|f(v_j)-f(v_{j+1})| : 3r-1 \le j \le 3r\} \\
&= \{|f(v_2)-f(v_3)|, |f(v_3)-f(v_4)|, \cdots, |f(v_{n-1})-f(v_n)|\} \\
&= \{|l_{m+4-3}-l_{m+6-3}|, |l_{m+6-3}-l_{m+8-3}|, |l_{m+10-6}-l_{m+12-6}|, |l_{m+12-6}-l_{m+14-6}|, \\
&\quad \cdots, |l_{m+2n-2-2n+1}-l_{m+2n-n+1}|\} \\
&= \{l_{m+2}, l_{m+4}, l_{m+5}, l_{m+7}, ..., l_{m+n}\}.
\end{aligned}$$

We find the edge labeling between the end vertex of r^{th} loop and the starting vertex of $(r+1)^{th}$ loop and $r = 1, 2, \cdots, \dfrac{n-2}{3}$. Let

$$\begin{aligned}
E_6' &= \bigcup_{r=1}^{\frac{n-2}{3}} \{f_1(v_{3r+1}\ v_{3r+2})\} = \bigcup_{r=1}^{\frac{n-2}{3}} \{|f(v_{3r+1})-f(v_{3r+2})|\} \\
&= \{|f(v_4)-f(v_5)|, |f(v_7)-f(v_8)|, \cdots, |f(v_{n-2})-f(v_{n-1})|\} \\
&= \{|l_{m+8-3}-l_{m+10-6}|, |l_{m+14-6}-l_{m+16-9}|, \cdots, |l_{m+2n-4-n+2}-l_{m+2n-2-n+1}|\} \\
&= \{|l_{m+5}-l_{m+4}|, |l_{m+8}-l_{m+7}|, \cdots, |l_{m+n-2}-l_{m+n}|\} \\
&= \{l_{m+3}, l_{m+6}, l_{m+9}, \cdots, l_{m+n-1}\}
\end{aligned}$$

Now, $E = \bigcup\limits_{i=1}^{6} E_i$ if $n \equiv 1(mod\ 3)$ and $E = \left(\bigcup\limits_{i=1}^{4} E_i\right) \bigcup E_5' \bigcup E_6'$ if $n \equiv 2(mod\ 3)$. So, the edges of G receive the distinct labels. Therefore, f is a Lucas graceful labeling. Hence, $G = C_m@P_n$ is a Lucas graceful graph when $m \equiv 0(mod\ 3)$ and $n \equiv 1, 2(mod\ 3)$. □

Example 2.14 The graph $C_9@P_7$ admits a Lucas graceful labeling, such as those shown in Fig.6.

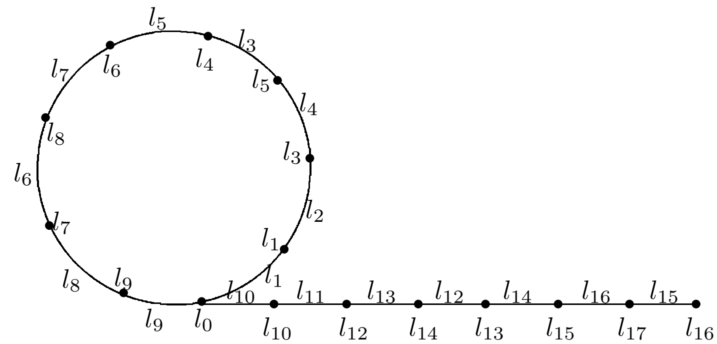

Fig.6

Definition 2.15 *The graph $K_{1,n} \odot 2P_m$ means that 2 copies of the path of length m is attached with each pendent vertex of $K_{1,n}$.*

Theorem 2.16 *The graph $K_{1,n} \odot 2P_m$ is a Lucas graceful graph.*

Proof Let $G = K_{1,n} \odot 2P_m$ with $V(G) = \{u_i : 0 \le i \le n\} \bigcup \left\{v_{ij}^{(1)}, v_{i,j}^{(2)} : 1 \le i \le n, 1 \le j \le m-1\right\}$ and $E(G) = \{u_0\ u_i : 1 \le i \le n\} \cup \left\{u_i\ v^{(1)_{i,j}}, u_i\ v_{i,j}^{(2)} : 1 \le i \le n \text{ and } 1 \le j \le m-1\right\} \cup \left\{v_{i,j}^{(1)}\ v_{i,j+1}^{(1)}, v_{i,j}^{(2)}\ v_{i,j+1}^{(2)} : 1 \le i \le n \text{ and } 1 \le j \le m-1\right\}$. Thus $|V(G)| = 2mn + n + 1$ and $|E(G)| = 2mn + n$.

For $i = 1, 2, \cdots, n$, define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}, a \in N$, by $f(u_0) = l_0$ $f(u_i) = l_{(2m+1)(i-1)+2}$; $f(v_{i,j}^{(1)}) = l_{(2m+1)(i-1)+2j+1}$, $1 \le j \le m$ and $f(v_{i,j}^{(2)}) = l_{(2m+1)(i-1)+2j+2}$, $1 \le j \le m$.

We claim that the edge labels are distinct. Let

$$\begin{aligned} E_1 &= \bigcup_{i=1}^{n} \{f_1(u_0\ u_i)\} = \bigcup_{i=1}^{n} \{|f(u_0) - f(u_i)|\} \\ &= \bigcup_{i=1}^{n} \left\{\left|l_0 - l_{(2m+1)(i-1)+2}\right|\right\} = \bigcup_{i=1}^{n} \left\{l_{(2m+1)(i-1)+2}\right\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \bigcup_{i=1}^{n} \left\{f_1(u_i v_{i,1}^{(1)}), f_1(u_i v_{i,1}^{(2)})\right\} \\ &= \bigcup_{i=1}^{n} \left\{\left|f(u_i) - f(v_{i,1}^{(1)}\right|, \left|f(u_i) - f(v_{i,1}^{(2)})\right|\right\} \end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{i=1}^{n}\left\{\left|l_{(2m+1)(i-1)+2}-l_{(2m+1)(i-1)+3}\right|,\left|l_{(2m+1)(i-1)+2}-l_{(2m+1)(i-1)+4}\right|\right\}\\
&= \bigcup_{i=1}^{n}\left\{l_{(2m+1)(i-1)+1},l_{(2m+1)(i-1)+3}\right\}\\
&= \{l_1,l_3\}\cup\{l_{2m+2},l_{2m+4}\}\cup\{l_{2mn+n-2m+1},l_{2mn+n-2m+3}\}\\
&= \{l_1,l_{2m+2},\cdots,l_{2mn+n-2m+1},l_3,l_{2m+4},\cdots,l_{2mn+n-2m+3}\},
\end{aligned}
$$

$$
\begin{aligned}
E_3 &= \bigcup_{i=1}^{n}\left\{\bigcup_{j=1}^{m-1}\left\{f_1(v_{i,j}^{(1)}\ v_{i,j+1}^{(1)}\right\}\right\}\\
&= \bigcup_{i=1}^{n}\left\{\bigcup_{j=1}^{m-1}\left\{\left|f(v_{i,j}^{(1)})-f(v_{i,j+1}^{(1)}\right|\right\}\right\}\\
&= \bigcup_{i=1}^{n}\left\{\bigcup_{j=1}^{m-1}\left\{\left|l_{(2m+1)(i-1)+2j+1}-l_{(2m+1)(i-1)+2j+3}\right|\right\}\right\}\\
&= \bigcup_{i=1}^{n}\left\{\bigcup_{j=1}^{m-1}\left\{l_{(2m+1)(i-1)+2j+2}\right\}\right\}\\
&= \bigcup_{i=1}^{n}\left\{l_{(2m+1)(i-1)+4},l_{(2m+1)(i-1)+6},\cdots,l_{(2m+1)(i-1)+2m}\right\}\\
&= \{l_4,l_6,\cdots,l_{2m}\}\cup\left\{l_{(2m+1)+4},l_{(2m+1)+6},\cdots,l_{(2m+1)(i-1)+2m}\right\}\bigcup\\
&\quad \cdots\bigcup\left\{l_{(2m+1)(n-1)+4},l_{(2m+1)(n-1)+6},\cdots,l_{(2m+1)(n-1)+2m}\right\}\\
&= \left\{l_4,\cdots,l_{2m},l_{2m+5},\cdots,l_{4m+1},\cdots,l_{(2m+1)(n-1)+4},l_{(2m+1)(n-1)+6},\cdots,l_{2mn+n-1}\right\},
\end{aligned}
$$

$$
\begin{aligned}
E_4 &= \bigcup_{i=1}^{n}\left\{\bigcup_{j=1}^{m-1}\left\{f_1(v_{i,j}^{(2)}\ v_{i,j+1}^{(2)}\right\}\right\}\\
&= \bigcup_{i=1}^{n}\left\{\bigcup_{j=1}^{m-1}\left\{\left|f(v_{i,j}^{(2)})-f(v_{i,j+1}^{(2)}\right|\right\}\right\}\\
&= \bigcup_{i=1}^{n}\left\{\bigcup_{j=1}^{m-1}\left\{\left|l_{(2m+1)(i-1)+2j+2}-l_{(2m+1)(i-1)+2j+4}\right|\right\}\right\}\\
&= \bigcup_{i=1}^{n}\left\{\bigcup_{j=1}^{m-1}\left\{l_{(2m+1)(i-1)+2j+3}\right\}\right\}\\
&= \bigcup_{i=1}^{n}\left\{l_{(2m+1)(i-1)+5},l_{(2m+1)(i-1)+7},\cdots,l_{(2m+1)(i-1)+2m+1}\right\}\\
&= \{l_5,\cdots,l_{2m+1}\}\bigcup\{l_{2m+1+5},l_{2m+1+7},\cdots,l_{2m+1+2m+1}\}\\
&\quad \bigcup\left\{l_{(2m+1)(n-1)+5},l_{(2m+1)(n-1)+7},\cdots,l_{(2m+1)(n-1)+(2m+1)}\right\}\\
&= \left\{l_5,\cdots,l_{2m+1},l_{2m+6},\cdots,l_{4m+1},\cdots,l_{(2m+1)(n-1)+5},l_{(2m+1(n-1)+7},\cdots,l_{(2m+1)+n}\right\}.
\end{aligned}
$$

Now, $E = \bigcup_{i=1}^{4} E_i = \{l_1, l_2, ..., l_{(2m+1)n}\}$. So, the edge labels of G are distinct. Therefore, f is a Lucas graceful labeling. Hence, $G = K_{1,n} \odot 2P_m$ is a Lucas graceful labeling. □

Example 2.17 The graph $K_{1,4} \odot 2P_4$ admits Lucas graceful labeling, such as those shown in Fig.7.

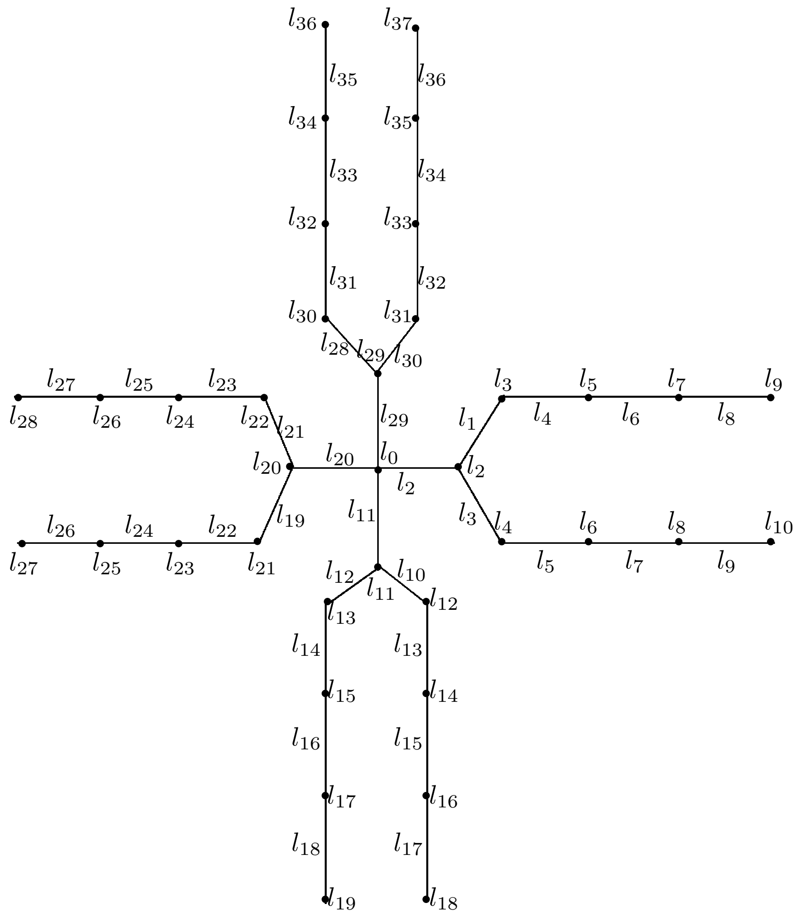

Fig.7

Theorem 2.18 *The graph $C_3@2P_n$ is Lucas graceful graph when $n \equiv 1(mod\ 3)$.*

Proof Let $G = C_3@2P_n$ with $V(G) = \{w_i : 1 \le i \le 3\} \cup \{u_i : 1 \le i \le n\} \cup \{v_i : 1 \le i \le n\}$ and the vertices w_2 and w_3 of C_3 are identified with v_1 and u_1 of two paths of length n respectively. Let $E(G) = \{w_i w_{i+1} : 1 \le i \le 2\} \cup \{u_i u_{i+1}, v_i v_{i+1} : 1 \le i \le n\}$ be the edge set of G. So, $|V(G)| = 2n+3$ and $|E(G)| = 2n+3$. Define $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_a\}, a \in N$ by $f(w_1) = l_{n+4}$; $f(u_i) = l_{n+3-i}, 1 \le i \le n+1$; $f(v_j) = l_{n+4+2j-3s}$, $3s-2 \le j \le 3s$ for $s = 1, 2, ..., \frac{n-1}{3}$ and $f(v_j) = l_{n+4+2j-3s}$ $3s-2 \le j \le 3s-1$ for $s = \frac{n-1}{3} + 1$.

We claim that the edge labels are distinct. Let

$$\begin{aligned}
E_1 &= \bigcup_{i=1}^{n}\{f_1(u_iu_{i+1}\} = \bigcup_{i=1}^{n}\{|f(u_i)-f(u_{i+1})|\}\\
&= \bigcup_{i=1}^{n}\{|l_{n+3-i}-l_{n+3-i-1}|\} = \bigcup_{i=1}^{n}\{|l_{n+3-i}-l_{n+2-i}|\}\\
&= \bigcup_{i=1}^{n}\{l_{n+1-i}\} = \{l_n, l_{n-1}, \cdots, l_1\},\\
E_2 &= \{f_1(u_1w_1), f_1(w_1v_1), f_1(v_1u_1)\}\\
&= \{|f(u_1)-f(w_1)|, |f(w_1-f(v_1)|, |f(v_1)-f(u_1)|\}\\
&= \{|l_{n+2}-l_{n+4}|, |l_{n+4}-l_{n+3}|, |l_{n+3}-l_{n+2}|\} = \{l_{n+3}, l_{n+2}, l_{n+1}\}.
\end{aligned}$$

For $s = 1, 2, \cdots, \dfrac{n-1}{3}$, let

$$\begin{aligned}
E_3 &= \bigcup_{s=1}^{\frac{n-1}{3}}\{f_1(v_jv_{j+1}) : 3s-2 \le j \le 3s-1\}\\
&= \bigcup_{s=1}^{\frac{n-1}{3}}\{|f(v_j)-f(v_{j+1})| : 3s-2 \le j \le 3s-1\}\\
&= \{|f(v_1)-f(v_2)|, |f(v_2)-f(v_3)|\} \cup \{|f(v_4)-f(v_5)|, |f(v_5)-f(v_6)|\} \bigcup\\
&\quad \cdots \bigcup \{|f(v_{n-3})-f(v_{n-2})|, |f(v_{n-2})-f(v_{n-1})|\}\\
&= \{|l_{n+3}-l_{n+5}|, |l_{n+5}-l_{n+7}|\} \cup \{|l_{n+6}-l_{n+8}|, |l_{n+8}-l_{n+10}|\} \bigcup\\
&\quad \cdots \bigcup \{|l_{2n-1}-l_{2n+1}|, |l_{2n+1}-l_{2n+3}|\}\\
&= \{l_{n+4}, l_{n+6}\} \bigcup \{l_{n+7}, l_{n+9}\} \bigcup \cdots \bigcup \{l_{2n}, l_{2n+2}\}.
\end{aligned}$$

We find the edge labeling between the end vertex of s^{th} loop and the starting vertex of $(s+1)^{th}$ loop and $1 \le s \le \dfrac{n-1}{3}$. Let

$$\begin{aligned}
E_4 &= \{f_1(v_jv_{j+1}) : j = 3s\} = \{|f(v_j)-f(v_{j+1})| : j = 3s\}\\
&= \{|f(v_3)-f(v_4)|, |f(v_6)-f(v_7)|, \cdots, |f(v_{n-1})-f(v_n)|\}\\
&= \{|l_{n+7}-l_{n+6}|, |l_{n+10}-l_{n+9}|, \cdots, |l_{2n+3}-l_{2n+2}|\} = \{l_5, l_8, \cdots, l_{2n+1}\}.
\end{aligned}$$

For $s = \dfrac{n-1}{3}+1$, let

$$\begin{aligned}
E_5 &= \{f_1(v_jv(_{j+1}) : j = 3s-2\} = \{|f(v_j)-f(v_{j+1})| : j = n\}\\
&= \{|f(v_n)-f(v_{n+1})|\} = \{|l_{n+4+2n-n-2}-l_{n+4+2n+2-n-2}|\}\\
&= \{|l_{2n+2}-l_{2n+4}|\} = \{l_{2n+3}\}.
\end{aligned}$$

Now, $E = \bigcup\limits_{s=1}^{5} E_i = \{l_1, l_2, ..., l_{2n+3}\}$. So, the edge labels of G are distinct. Therefore, f is a Lucas graceful labeling. Hence, $G = C_3@2P_n$ is a Lucas graceful graph if $n \equiv 1(mod\ 3)$. $\Box$

Example 2.19 The graph $C_3@2P_4$ admits Lucas graceful labeling shown in Fig.8.

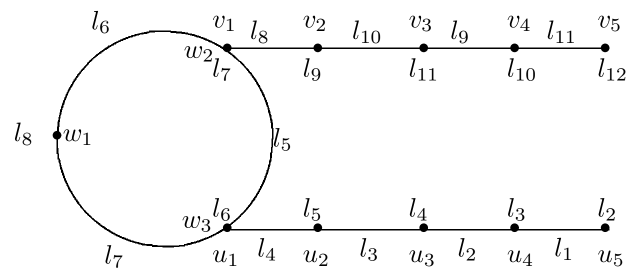

Fig.8

Theorem 2.20 *The graph $C_n@K_{1,2}$ is a Lucas graceful graph if $n \equiv 1(mod\ 3)$.*

Proof Let $G = C_n@K_{1,2}$ with $V(G) = \{u_i : 1 \le i \le n\} \cup \{v_1, v_2\}$, $E(G) = \{u_i u_{i+1} : 1 \le i \le n-1\} \cup \{u_n u_1, u_n v_n, u_n v_2\}$. So, $|V(G)| = n+2$ and $|E(G)| = n+2$. Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_a\}, a \ \epsilon\ N$ by $f(u_1) = 0$, $f(v_1) = l_n$, $f(v_2) = l_{n+3}$; $f(u_i) = l_{2i-3s}$, $3s-1 \le i \le 3s+1$ for $s = 1, 2, ..., \dfrac{n-4}{3}$ and $f(u_i) = l_{2i-3s}$, $3s-1 \le i \le 3s$ for $s = \dfrac{n-1}{3}$. We claim that the edge labels are distinct. Let

$$\begin{aligned}
E_1 &= \{f_1(u_1u_2), f_1(u_nv_1), f_1(u_nv_2), f_1(u_nu_1)\} \\
&= \{|f(u_1) - f(u_2)|, |f(u_n) - f(v_1)|, |f(u_n) - f(v_2)|, |f(u_n) - f(v_1)|\} \\
&= \{|l_0 - l_1|, |l_{n+1} - l_n|, |l_{n+1} - l_{n+3}|, |l_{n+1} - l_0|\} \\
&= \{l_1, l_{n-1}, l_{n+2}, l_{n+1}\},
\end{aligned}$$

$$\begin{aligned}
E_2 &= \bigcup_{s=1}^{\frac{n-4}{3}} \{f_1(u_iu_{i+1}) : 3s-1 \le i \le 3s\} \\
&= \bigcup_{s=1}^{\frac{n-4}{3}} \{|f(u_i) - f(u_{i+1})| : 3s-1 \le i \le 3s\} \\
&= \{|f(u_2) - f(u_3)|, |f(u_3) - f(u_4)|\} \bigcup \{|f(u_5) - f(u_6)|, |f(u_6) - f(u_7)|\} \bigcup \\
&\quad \cdots \bigcup \{|f(u_{n-5}) - f(u_{n-4})|, |f(u_{n-4}) - f(u_{n-3})|\} \\
&= \{|l_1 - l_3|, |l_3 - l_5|\} \bigcup \{|l_4 - l_6|, |l_6 - l_8|\} \bigcup \\
&\quad \cdots \bigcup \{|l_{n-6} - l_{n-4}|, |l_{n-5} - l_{n-2}|\} \\
&= \{l_2, l_4\} \bigcup \{l_5, l_7\} \bigcup \cdots \bigcup \{l_{n-5}, l_{n-3}\} = \{l_2, l_4, l_5, l_7, \cdots, l_{n-5}, l_{n-3}\}
\end{aligned}$$

We determine the edge labeling between the end vertex of s^{th} loop and the starting vertex

of $(s+1)^{th}$ loop and $1 \leq s \leq \dfrac{n-4}{3}$. Let

$$\begin{aligned} E_3 &= \{f_1(u_iu_{i+1}) : i = 3s+1\} = \{|f(u_i) - f(u_{i+1})| : i = 3s+1\} \\ &= \{|f(u_4) - f(u_5)|, |f(u_7) - f(u_8)|, \cdots, |f(u_{n-3}) - f(u_{n-2})|\} \\ &= \{|l_{8-3} - l_{10-6}|, |l_{14-6} - l_{16-9}|, \cdots, |l_{2n-6-n+4} - l_{2n-4-n+1}|\} \\ &= \{|l_5 - l_4|, |l_8 - l_7|, \cdots, |l_{n-2} - l_{n-3}|\} = \{l_3, l_6, \cdots, l_{n-4}\}. \end{aligned}$$

For $s = \frac{n-1}{3}$, let

$$\begin{aligned} E_4 &= \{f_1(u_iu_{i+1}) : 3s-1 \leq i \leq 3s\} \\ &= \{|f(u_i) - f(u_{i+1})| : 3s-1 \leq i \leq 3s\} \\ &= \{|f(u_{n-2}) - f(u_{n-1})|, |f(u_{n-1}) - f(u_n)|\} \\ &= \{|l_{2n-4-n+1} - l_{2n-2-n+1}|, |l_{2n-2-n+1} - l_{2n-n+1}|\} \\ &= \{|l_{n-3} - l_{n-1}|, |l_{n-1} - l_{n+1}|\} = \{l_{n-2}, l_n\} \end{aligned}$$

Now, $E = \bigcup\limits_{i=1}^{4} E_i = \{l_1, l_2, ..., l_{n+2}\}$. So, the edge labels of G are distinct. Therefore, f is a Lucas graceful labeling. Hence, $G = C_n@K_{1,2}$ is a Lucas graceful graph. □

Example 2.21 The graph $C_{10}@K_{1,2}$ admits Lucas graceful labeling shown in Fig.9.

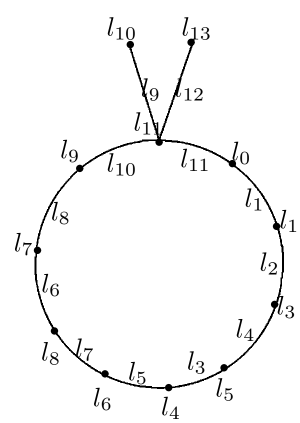

Fig.9

§3. Strong Lucas Graceful Graphs

In this section, we prove that the graphs $K_{1,n}$ and F_n admit strong Lucas graceful labeling.

Definition 3.1 *Let G be a (p,q) graph. An injective function $f : V(G) \to \{l_0, l_1, l_2, \cdots, l_q\}$ is said to be strong Lucas graceful labeling if an induced edge labeling $f_1(uv) = |f(u) - f(v)|$ is a bijection on to the set $\{l_1, l_2, ..., l_q\}$ with the assumption of $l_0 = 0, l_1 = 1, l_2 = 3, l_3 = 4, l_4 =$*

$7, l_5 = 11, \cdots,$. *Then G is called strong Lucas graceful graph if it admits strong Lucas graceful labeling.*

Theorem 3.2 *The graph $K_{1,n}$ is a strong Lucas graceful graph.*

Proof Let $G = K_{1,n}$ and $V = V_1 \cup V_2$ be the bipartition of $K_{1,n}$ with $V_1 = \{u_1\}$ and $V_2 = \{u_1, u_2, ..., u_n\}$. Then, $|V(G)| = n+1$ and $|E(G)| = n$. Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_n\}$ by $f(u_0) = l_0$, $f(u_1) = l_1, 1 \leq i \leq n$. We claim that the edge labels are distinct. Notice that

$$\begin{aligned} E &= \{f_1(u_0u_1) : 1 \leq i \leq n\} = \{f(u_0) - f(u_1) : 1 \leq i \leq n\} \\ &= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_2)|, ..., |f(u_0) - f(u_n)|\} \\ &= \{|l_0 - l_1|, |l_0 - l_2|, ..., |l_0 - l_n|\} = \{l_1, l_2, ..., l_n\} \end{aligned}$$

So, the edges of G receive the distinct labels. Therefore, f is a strong Lucas graceful labeling. Hence, K_1, n the path is a strong Lucas graceful graph. □

Example 3.3 The graph $K_{1,9}$ admits strong Lucas graceful labeling shown in Fig.10.

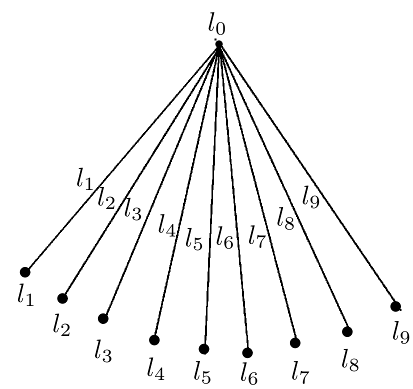

Fig.10

Definition 3.4([2]) *Let $u_1, u_2, ..., u_n, u_{n+1}$ be the vertices of a path and u_0 be a vertex which is attached with $u_1, u_2, ..., u_n, u_{n+1}$. Then the resulting graph is called Fan and is denoted by $F_n = P_n + K_1$.*

Theorem 3.5 *The graph $F_n = P_n + K_1$ is a Lucas graceful graph.*

Proof Let $G = F_n$ and $u_1, u_2, ..., u_n, u_{n+1}$ be the vertices of a path P_n with the central vertex u_0 joined with $u_1, u_2, ..., u_n, u_{n+1}$. Clearly, $|V(G)| = n + 2$ and $|E(G)| = 2n + 1$. Define $f : V(G) \to \{l_0, l_1, l_2, ..., l_{2n+1}\}$ by $f(u_0) = l_0$ and $f(u_i) = l_{2i-1}, 1 \leq i \leq n + 1$. We claim that the edge labels are distinct.

Calculation shows that

$$\begin{aligned} E_1 &= \{f_1(u_iu_{i+1}) : 1 \leq i \leq n\} = \{|f_(u_i) - f(u_{i+1})| : 1 \leq i \leq n\} \\ &= \{|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|, ..., |f(u_n) - f(u_{n+1})|\} \\ &= \{|l_1 - l_3|, |l_3 - l_5|, ..., |l_{2n-1} - l_{2n+1}|\} = \{l_2, l_4, ..., l_{2n}\}, \end{aligned}$$

$$\begin{aligned} E_2 &= \{f_1(u_0u_i) : 1 \leq i \leq n+1\} = \{|f(u_0) - f(u_i)| : 1 \leq i \leq n+1\} \\ &= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_2)|, ..., |f(u_0) - f(u_{n+1})|\} \\ &= \{|l_0 - l_1|, |l_0 - l_3|, ..., |l_0 - l_{2n+1}|\} = \{l_1, l_3, ..., l_{2n+1}\}. \end{aligned}$$

Whence, $E = E_1 \cup E_2 = \{l_1, l_2, ..., l_{2n}, l_{2n+1}\}$. Thus the edges of F_n receive the distinct labels. Therefore, f is a Lucas graceful labeling. Consequently, $F_n = P_n + K_1$ is a Lucas graceful graph. □

Example 3.6 The graph $F_7 = P_7 + K_1$ admits Lucas graceful graph shown in Fig.11.

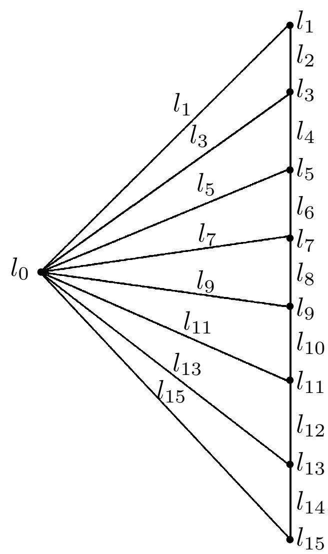

Fig.11

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Sequences on Graphs with Symmetries

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Abstract: An interesting symmetry on multiplication of numbers found by Prof.Smarandache recently. By considering integers or elements in groups on graphs, we extend this symmetry on graphs and find geometrical symmetries. For extending further, Smarandache's or combinatorial systems are also discussed in this paper, particularly, the CC conjecture presented by myself six years ago, which enables one to construct more symmetrical systems in mathematical sciences.

Key Words: Smarandache sequence, labeling, Smarandache beauty, graph, group, Smarandache system, combinatorial system, CC conjecture.

AMS(2010): 05C21, 05E18

§1. Sequences

Let $\mathbb{Z}^+$ be the set of non-negative integers and Γ a group. We consider sequences $\{i(n)|n \in \mathbb{Z}^+\}$ and $\{g_n \in \Gamma|n \in \mathbb{Z}^+\}$ in this paper. There are many interesting sequences appeared in literature. For example, the sequences presented by Prof.Smarandache in references [2], [13] and [15] following:

(1) **Consecutive sequence**

$1, 12, 123, 1234, 12345, 123456, 1234567, 12345678, \cdots;$

(2) **Digital sequence**

$1, 11, 111, 1111, 11111, 11111, 1111111, 11111111, \cdots$

(3) **Circular sequence**

$1,\ \ 12, 21,\ \ 123, 231, 312,\ \ 1234, 2341, 3412, 4123, \cdots;$

(4) **Symmetric sequence**

$1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 123454321, 1234554321, \cdots;$

(5) **Divisor product sequence**

[1]Reported at *The 7th Conference on Number Theory and Smarandache's Notion, Xian, P.R.China*

[2]Received December 18, 2010. Accepted February 18, 2011.

$1, 2, 3, 8, 5, 36, 7, 64, 27, 100, 11, 1728, 13, 196, 225, 1024, 17, 5832, 19, \cdots;$

(6) **Cube-free sieve**

$2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 25, 26, 28, 29, 30, \cdots.$

He also found three nice symmetries for these integer sequences recently.

First Symmetry

$$\begin{aligned}
1 \times 8 + 1 &= 9 \\
12 \times 8 + 2 &= 98 \\
123 \times 8 + 3 &= 987 \\
1234 \times 8 + 4 &= 9876 \\
12345 \times 8 + 5 &= 98765 \\
123456 \times 8 + 6 &= 987654 \\
1234567 \times 8 + 7 &= 9876543 \\
12345678 \times 8 + 8 &= 98765432 \\
123456789 \times 8 + 9 &= 987654321
\end{aligned}$$

Second Symmetry

$$\begin{aligned}
1 \times 9 + 2 &= 11 \\
12 \times 9 + 3 &= 111 \\
123 \times 9 + 4 &= 1111 \\
1234 \times 9 + 5 &= 11111 \\
12345 \times 9 + 6 &= 111111 \\
123456 \times 9 + 7 &= 1111111 \\
1234567 \times 9 + 8 &= 11111111 \\
12345678 \times 9 + 9 &= 111111111 \\
123456789 \times 9 + 10 &= 1111111111
\end{aligned}$$

Third Symmetry

$$\begin{aligned}
1 \times 1 &= 1 \\
11 \times 11 &= 121 \\
111 \times 111 &= 12321 \\
1111 \times 1111 &= 1234321 \\
11111 \times 11111 &= 12345431
\end{aligned}$$

$$\begin{aligned} 111111 \times 111111 &= 12345654321 \\ 1111111 \times 1111111 &= 1234567654321 \\ 11111111 \times 11111111 &= 13456787654321 \\ 111111111 \times 111111111 &= 12345678987654321 \end{aligned}$$

Notice that a Smarandache sequence is not closed under operation, but a group is, which enables one to get symmetric figure in geometry. Whence, we also consider labelings on graphs G by that elements of groups in this paper.

§2. Graphs with Labelings

A *graph* G is an ordered 3-tuple $(V(G), E(G); I(G))$, where $V(G), E(G)$ are finite sets, called vertex and edge set respectively, $V(G) \neq \emptyset$ and $I(G) : E(G) \to V(G) \times V(G)$. Usually, the cardinality $|V(G)|$ is called the order and $|E(G)|$ the size of a graph G.

A graph $H = (V_1, E_1; I_1)$ is a *subgraph* of a graph $G = (V, E; I)$ if $V_1 \subseteq V$, $E_1 \subseteq E$ and $I_1 : E_1 \to V_1 \times V_1$, denoted by $H \subset G$.

Example 2.1 A graph G is shown in Fig.2.1, where, $V(G) = \{v_1, v_2, v_3, v_4\}$, $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$ and $I(e_i) = (v_i, v_i), 1 \leq i \leq 4; I(e_5) = (v_1, v_2) = (v_2, v_1), I(e_8) = (v_3, v_4) = (v_4, v_3), I(e_6) = I(e_7) = (v_2, v_3) = (v_3, v_2)$, $I(e_8) = I(e_9) = (v_4, v_1) = (v_1, v_4)$.

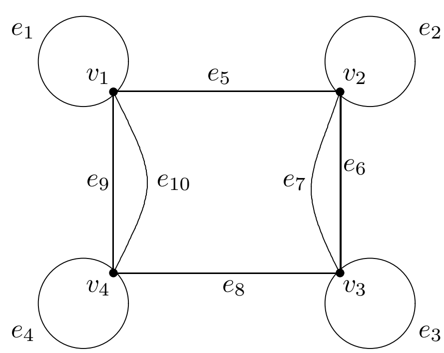

Fig. 2.1

An automorphism of a graph G is a $1-1$ mapping $\theta : V(G) \to V(G)$ such that

$$\theta(u, v) = (\theta(u), \theta(v)) \in E(G)$$

holds for $\forall (u, v) \in E(G)$. All such automorphisms of G form a group under composition operation, denoted by Aut$G$. A graph G is *vertex-transitive* if Aut$G$ is transitive on $V(G)$.

A *graph family* $\mathscr{F}_P$ is the set of graphs whose each element possesses a graph property P. Some well-known graph families are listed following.

Walk. A walk of a graph G is an alternating sequence of vertices and edges $u_1, e_1, u_2, e_2, \cdots, e_n, u_{n_1}$ with $e_i = (u_i, u_{i+1})$ for $1 \leq i \leq n$.

Path and Circuit. A walk such that all the vertices are distinct and a circuit or a cycle is such a walk $u_1, e_1, u_2, e_2, \cdots, e_n, u_{n_1}$ with $u_1 = u_n$ and distinct vertices. A graph $G = (V, E; I)$ is *connected* if there is a path connecting any two vertices in this graph.

Tree. A tree is a connected graph without cycles.

n-Partite Graph. A graph G is n-partite for an integer $n \geq 1$, if it is possible to partition $V(G)$ into n subsets $V_1, V_2, \cdots, V_n$ such that every edge joints a vertex of $V_i$to a vertex of V_j, $j \neq i$, $1 \leq i, j \leq n$. A *complete n-partite graph G* is such an n-partite graph with edges $uv \in E(G)$ for $\forall u \in V_i$ and $v \in V_j$ for $1 \leq i, j \leq n$, denoted by $K(p_1, p_2, \cdots, p_n)$ if $|V_i| = p_i$ for integers $1 \leq i \leq n$. Particularly, if $|V_i| = 1$ for integers $1 \leq i \leq n$, such a complete n-partite graph is called *complete graph* and denoted by K_n.

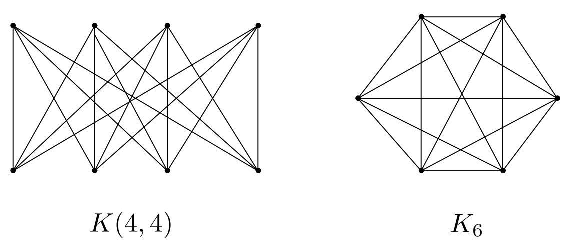

Fig.2.2

Two operations of graphs used in this paper are defined as follows:

Cartesian Product. A Cartesian product $G_1 \times G_2$ of graphs G_1 with G_2 is defined by $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 \times G_2$ are adjacent if and only if either $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$ or $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$.

The graph $K_2 \times P_6$ is shown in Fig.2.3 following.

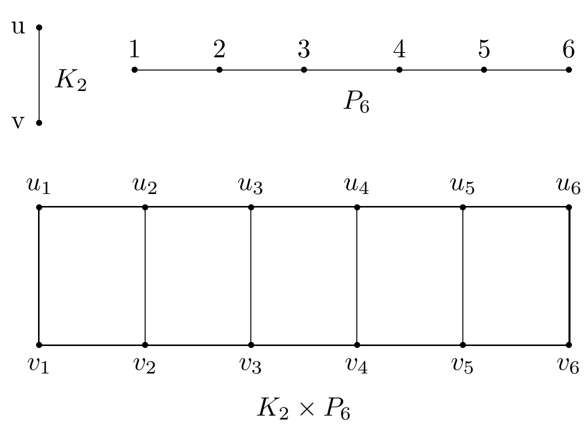

Fig.2.3

Union. The *union* $G\cup H$ of graphs G and H is a graph $(V(G\cup H), E(G\cup H), I(G\cup H))$ with

$$V(G\cup H)=V(G)\cup V(H), \quad E(G\cup H)=E(G)\cup E(H) \quad \text{and} \quad I(G\cup H)=I(G)\cup I(H).$$

Labeling. Now let G be a graph and $N\subset \mathbb{Z}^+$. A *labeling* of G is a mapping $l_G: V(G)\cup E(G)\to N$ with each labeling on an edge (u,v) is induced by a ruler $r(l_G(u), l_G(v))$ with additional conditions.

Classical Labeling Rulers. The following rulers are usually found in literature.

Ruler R1. $r(l_G(u), l_G(v)) = |l_G(u)-l_G(v)|$.

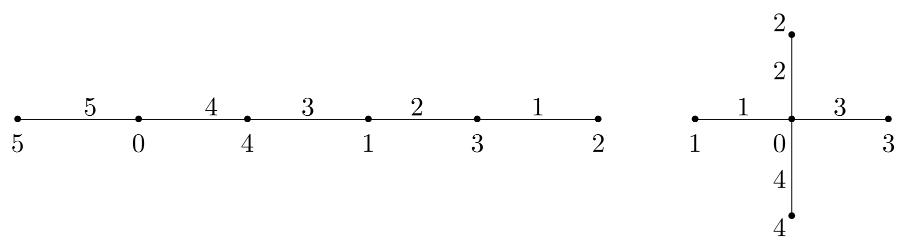

Fig.2.4

Such a labeling l_G is called to be a graceful labeling of G if $l_G(V(G))\subset\{0,1,2,\cdots,|V(G)|\}$ and $l_G(E(G))=\{1,2,\cdots,|E(G)|\}$. For example, the graceful labelings of P_6 and $S_{1.4}$ are shown in Fig.2.4.

Graceful Tree Conjecture (A.Rose, 1966) *Any tree is graceful.*

There are hundreds papers on this conjecture. But it is opened until today.

Ruler R2. $r(l_G(u), l_G(v)) = l_G(u)+l_G(v)$.

Such a labeling l_G on a graph G with q edges is called to be harmonious on G if $l_G(V(G))\subset \mathbb{Z}(\text{mod}q)$ such that the resulting edge labels $l_G(E(G))=\{1,2,\cdots,|E(G)|\}$ by the induced labeling $l_G(u,v)=l_G(u)+l_G(v)$ (mod$q$) for $\forall(u,v)\in E(G)$. For example, ta harmonious labeling of P_6 are shown in Fig.2.5 following.

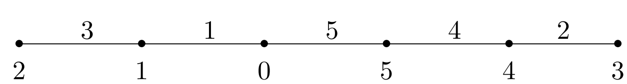

Fig.2.5

Update results on classical labeling on graphs can be found in a survey paper [4] of Gallian.

Smarandachely Labeling Rulers. There are many new labelings on graphs appeared in *International J.Math.Combin.* in recent years. Such as those shown in the following.

Ruler R3. A *Smarandachely k-constrained labeling* of a graph $G(V,E)$ is a bijective mapping $f: V\cup E\to\{1,2,..,|V|+|E|\}$ with the additional conditions that $|f(u)-f(v)|\geq k$ whenever

$uv \in E$, $|f(u) - f(uv)| \geq k$ and $|f(uv) - f(vw)| \geq k$ whenever $u \neq w$, for an integer $k \geq 2$. A graph G which admits a such labeling is called a Smarandachely k-constrained total graph, abbreviated as $k - CTG$. An example for $k = 5$ on P_7 is shown in Fig.2.6.

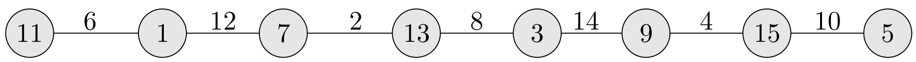

Fig.2.6

The minimum positive integer n such that the graph $G \cup \overline{K}_n$ is a $k - CTG$ is called *k-constrained number* of the graph G and denoted by $t_k(G)$, the corresponding labeling is called a minimum k-constrained total labeling of G. Update results for $t_k(G)$ in [3] and [12] are as follows:

(1) $t_2(P_n)=\begin{cases} 2 \ \ if \ \ n = 2, \\ 1 \ \ if \ \ n = 3, \\ 0 \ \ else. \end{cases}$

(2) $t_2(C_n) = 0$ if $n \geq 4$ and $t_2(C_3) = 2$.

(3) $t_2(K_n) = 0$ if $n \geq 4$.

(4) $t_2(K(m,n))=\begin{cases} 2 \ \ if \ \ n = 1 \ \ and \ \ m = 1, \\ 1 \ \ if \ \ n = 1 \ \ and \ \ m \geq 2, \\ 0 \ \ else. \end{cases}$

(5) $t_k(P_n)=\begin{cases} 0 \ \ if \ \ k \leq k_0, \\ 2(k - k_0) - 1 \ \ if \ \ k > k_0 \ \ and \ \ 2n \equiv 0 (mod\ 3), \\ 2(k - k_0) \ \ if \ k > k_0 \ \ and \ \ 2n \equiv 1 \ or \ 2 (mod\ 3). \end{cases}$

(6) $t_k(C_n) = \begin{cases} 0 \ \ if \ \ k \leq k_0, \\ 2(k - k_0) \ \ if \ \ k > k_0 \ \ and \ \ 2n \equiv 0 \ (mod\ 3), \\ 3(k - k_0) \ \ if \ k > k_0 \ \ and \ \ 2n \equiv 1 \ or \ 2 (mod\ 3), \end{cases}$

where $k_0 = \lfloor \frac{2n-1}{3} \rfloor$. More results on $t_k(G)$ cam be found in references.

Ruler R4. Let G be a graph and $f : V(G) \to \{1, 2, 3, \cdots, |V| + |E(G)|\}$ be an injection. For each edge $e = uv$ and an integer $m \geq 2$, the induced *Smarandachely edge m-labeling* f_S^* is defined by

$$f_S^*(e) = \left\lceil \frac{f(u) + f(v)}{m} \right\rceil.$$

Then f is called a *Smarandachely super m-mean labeling* if $f(V(G)) \cup \{f^*(e) : e \in E(G)\} = \{1, 2, 3, \cdots, |V| + |E(G)|\}$. A graph that admits a Smarandachely super mean m-labeling is called Smarandachely super m-mean graph. Particularly, if $m = 2$, we know that

$$f^*(e) = \begin{cases} \dfrac{f(u) + f(v)}{2} & \text{if } f(u) + f(v) \text{ is even;} \\ \dfrac{f(u) + f(v) + 1}{2} & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

A Smarandache super 2-mean labeling on P_6^2 is shown in Fig.2.7.

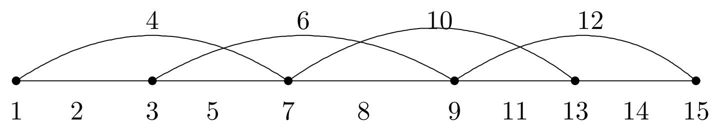

Fig.2.7

Now we have know graphs P_n, C_n, K_n,K(2,n), $(n \geq 4)$, $K(1,n)$ for $1 \leq n \leq 4$, $C_m \times P_n$ for $n \geq 1, m = 3, 5$ have Smarandachely super 2-mean labeling. More results on Smarandachely super m-mean labeling of graphs can be found in references in [1], [11], [17] and [18].

§3. Smarandache Sequences on Symmetric Graphs

Let $l_G^S : V(G) \rightarrow \{1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111\}$ be a vertex labeling of a graph G with edge labeling $l_G^S(u, v)$ induced by $l_G^S(u)l_G^S(v)$ for $(u, v) \in E(G)$ such that $l_G^S(E(G)) = \{1, 121, 12321, 1234321, 123454321, 12345654321, 1234567654321, 123456787654321,$ $12345678987654321\}$, i.e., $l_G^S(V(G) \cup E(G))$ contains all numbers appeared in the Smarandachely third symmetry. Denote all graphs with l_G^S labeling by $\mathscr{L}^S$. Then it is easily find a graph with a labeling l_G^S in Fig.3.1 following.

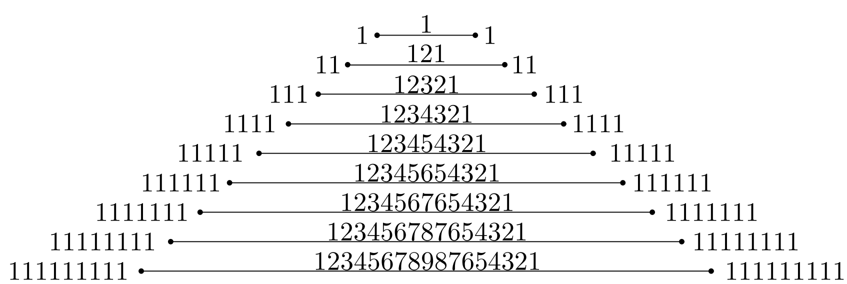

Fig.3.1

Generally, we know the following result.

Theorem 3.1 *Let $G \in \mathscr{L}^S$. Then $G = \bigcup_{i=1}^{n} H_i$ for an integer $n \geq 9$, where each H_i is a connected graph. Furthermore, if G is vertex-transitive graph, then $G = nH$ for an integer $n \geq 9$, where H is a vertex-transitive graph.*

Proof Let $C(i)$ be the connected component with a label i for a vertex u, where $i \in \{1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111\}$. Then all vertices v in $C(i)$ must be with label $l_G^S(v) = i$. Otherwise, if there is a vertex v with $l_G^S(v) = j \in \{1, 11, 111, 1111, 11111,$ $111111, 1111111, 11111111, 111111111\} \setminus \{i\}$, let $P(u, v)$ be a path connecting vertices u and v.

Then there must be an edge (x, y) on $P(u, v)$ such that $l_G^S(x) = i$, $l_G^S(y) = j$. By definition, $i \times j \notin l_G^S(E(G))$, a contradiction. So there are at least 9 components in G.

Now if G is vertex-transitive, we are easily know that each connected component $C(i)$ must be vertex-transitive and all components are isomorphic. □

The smallest graph in $\mathscr{L}_v^S$ is the graph $9K_2$ shown in Fig.3.1. It should be noted that each graph in $\mathscr{L}_v^S$ is not connected. For finding a connected one, we construct a graph $\widehat{Q}_k$ following on the digital sequence

$$1, 11, 111, 1111, 11111, \cdots, \underbrace{11\cdots 1}_{k}.$$

by

$$V(\widetilde{Q}_k) = \{1, 11, \cdots, \underbrace{11\cdots 1}_{k}\} \bigcup \{1', 11', \cdots, \underbrace{11\cdots 1'}_{k}\},$$

$$E(\widetilde{Q}_k) = \{(1, \underbrace{11\cdots 1}_{k}), (x, x'), (x, y) | x, y \in V(\widetilde{Q}) \text{ differ in precisely one } 1\}.$$

Now label $x \in V(\widetilde{Q})$ by $l_G(x) = l_G(x') = x$ and $(u, v) \in E(\widetilde{Q})$ by $l_G(u)l_G(v)$. Then we have the following result for the graph $\widetilde{Q}_k$.

Theorem 3.2 *For any integer $m \geq 3$, the graph $\widetilde{Q}_m$ is a connected vertex-transitive graph of order $2m$ with edge labels*

$$l_G(E(\widetilde{Q})) = \{1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 12345431, \cdots\},$$

i.e., the Smarandache symmetric sequence.

Proof Clearly, $\widetilde{Q}_m$ is connected. We prove it is a vertex-transitive graph. For simplicity, denote $\underbrace{11\cdots 1}_{i}$, $\underbrace{11\cdots 1'}_{i}$ by $\overline{i}$ and $\overline{i}'$, respectively. Then $V(\widetilde{Q}_m) = \{\overline{1}, \overline{2}, \cdots, \overline{m}\}$. We define an operation $+$ on $V(\widetilde{Q}_k)$ by

$$\overline{k} + \overline{l} = \underbrace{11\cdots 1}_{k+l(\bmod k)} \quad \text{and} \quad \overline{k}' + \overline{l}' = \overline{k+l}', \quad \overline{k}'' = \overline{k}$$

for integers $1 \leq k, l \leq m$. Then an element $\overline{i}$ naturally induces a mapping

$$i^* : \ \overline{x} \to \overline{x+i}, \quad \text{for} \ \ \overline{x} \in V(\widetilde{Q}_m).$$

It should be noted that i^* is an automorphism of $\widetilde{Q}_m$ because tuples $\overline{x}$ and $\overline{y}$ differ in precisely one 1 if and only if $\overline{x+i}$ and $\overline{y+i}$ differ in precisely one 1 by definition. On the other hand, the mapping $\tau : \overline{x} \to \overline{x}'$ for $\forall \overline{x} \in$ is clearly an automorphism of $\widetilde{Q}_m$. Whence,

$$\mathscr{G} = \langle \ \tau, i^* \ | \ 1 \leq i \leq m\rangle \preceq \mathrm{Aut}\widetilde{\mathrm{Q}}_\mathrm{m},$$

which acts transitively on $V(\widetilde{Q})$ because $(\overline{y-x})^*(\overline{x}) = \overline{y}$ for $\overline{x}, \overline{y} \in V(\widetilde{Q}_m)$ and $\tau : \overline{x} \to \overline{x}'$.

Calculation shows easily that

$$l_G(E(\widetilde{Q}_m)) = \{1, 11, 121, 1221, 12321, 123321, 1234321, 12344321, 12345431, \cdots\},$$

i.e., the Smarandache symmetric sequence. This completes the proof. □

By the definition of graph $\widetilde{Q}_m$, w consequently get the following result by Theorem 3.2.

Corollary 3.3 *For any integer* $m \geq 3$, $\widetilde{Q}_m \simeq C_m \times P_2$.

The smallest graph containing the third symmetry is $\widetilde{Q}_9$ shown in Fig.3.2 following,

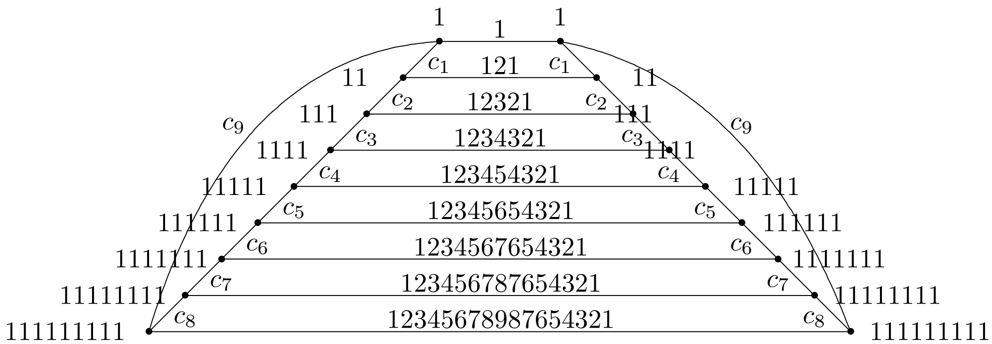

Fig.3.2

where $c_1 = 11$, $c_2 = 1221$, $c_3 = 123321$, $c_4 = 12344321$, $c_5 = 12344321$, $c_5 = 1234554321$, $c_6 = 123456654321$, $c_7 = 12345677654321$, $c_8 = 1234567887654321$, $c_9 = 123456789987654321$.

§4. Groups on Symmetric Graphs

In fact, the Smarandache digital or symmetric sequences are subsequences of $\mathbb{Z}$, a special infinite Abelian group. We consider generalized labelings on vertex-transitive graphs following.

Problem 4.1 *Let* $(\Gamma; \circ)$ *be an Abelian group generated by* $x_1, \cdots, x_n$. *Thus* $\Gamma = \langle x_1, x_2, \cdots, x_n | W_1, \cdots \rangle$. *Find connected vertex-transitive graphs* G *with a labeling* $l_G : V(G) \rightarrow \{1_\Gamma, x_1, x_2, \cdots, x_n\}$ *and induced edge labeling* $l_G(u, v) = l_G(u) \circ l_G(v)$ *for* $(u, v) \in E(G)$ *such that*

$$l_G(E(G)) = \{1_\Gamma, x_1^2, x_1 \circ x_2, x_2^2, x_2 \circ x_3, \cdots, x_{n-1} \circ x_n, x_n^2\}.$$

Similar to that of Theorem 3.2, we know the following result.

Theorem 4.2 *Let* $(\Gamma; \circ)$ *be an Abelian group generated by* $x_1, x_2, \cdots, x_n$ *for an integer* $n \geq 1$. *Then there are vertex-transitive graphs* G *with a labeling* $l_G : V(G) \rightarrow \{1_\Gamma, x_1, x_2, \cdots, x_n\}$ *such that the induced edge labeling by* $l_G(u, v) = l_G(u) \circ l_G(v)$ *with*

$$l_G(E(G)) = \{1_\Gamma, x_1^2, x_1 \circ x_2, x_2^2, x_2 \circ x_3, \cdots, x_{n-1} \circ x_n, x_n^2\}.$$

Proof For any integer $m \geq 1$, define a graph $\widehat{Q}_{m,n,k}$ by

$$V(\widehat{Q}_{m,n,k}) = \left(\bigcup_{i=0}^{m-1} U^{(i)}[x]\right) \bigcup \left(\bigcup_{i=0}^{m-1} W^{(i)}[y]\right) \bigcup \cdots \bigcup \left(\bigcup_{i=0}^{m-1} U^{(i)}[z]\right)$$

where $|\{U^{(i)}[x], v^{(i)}[y], \cdots, W^{(i)}[z]\}| = k$ and

$$\begin{aligned} U^{(i)}[x] &= \{x_0^{(i)}, x_1^{(i)}, x_2^{(i)}, \cdots, x_n^{(i)}\}, \\ V^{(i)}[y] &= \{(y_0)^{(i)}, y_1^{(i)}, y_2^{(i)}, \cdots, y_n^{(i)}\}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ W^{(i)}[z] &= \{(z_0)^{(i)}, z_1^{(i)}, z_2^{(i)}, \cdots, z_n^{(i)}\} \end{aligned}$$

for integers $0 \le i \le m-1$, and

$$E(\widehat{Q}_{m,n}) = E_1 \bigcup E_2 \bigcup E_3,$$

where $E_1 = \{\ (x_l^{(i)}, y_l^{(i)}), \cdots, (z_l^{(i)}, x_l^{(i)}) \mid 0 \le l \le n-1,\ 0 \le i \le m-1\}$, $E_2 = \{\ (x_l^{(i)}, x_{l+1}^{(i)}),$ $(y_l^{(i)}, y_{l+1}^{(i)}), \cdots, (z_l^{(i)}, z_{l+1}^{(i)}) \mid 0 \le l \le n-1,\ 0 \le i \le m-1,$ where $l+1 \equiv (\mathrm{mod} n)\}$ and $E_3 = \{(x_l^{(i)}, x_l^{(i+1)}),\ (y_l^{(i)}, y_l^{(i+1)}), \cdots, (z_l^{(i)}, z_l^{(i+1)}) | 0 \le l \le n-1,\ 0 \le i \le m-1,$ where $i+1 \equiv (\mathrm{mod} m)\}$. Then is clear that $\widehat{Q}_{m,n,k}$ is connected. We prove this graph is vertex-transitive. In fact, by defining three mappings

$$\begin{aligned} &\theta : x_l^{(i)} \to x_{l+1}^{(i)},\ y_l^{(i)} \to y_{l+1}^{(i)}, \cdots, z_l^{(i)} \to z_{l+1}^{(i)}, \\ &\tau : x_l^{(i)} \to y_l^{(i)}, \cdots, z_l^{(i)} \to x_l^{(i)}, \\ &\sigma : x_l^{(i)} \to x_l^{(i+1)},\ y_l^{(i)} \to y_l^{(i+1)}, \cdots, z_l^{(i)} \to z_l^{(i+1)}, \end{aligned}$$

where $1 \le l \le n,\ 1 \le i \le m,\ i+1(\mathrm{mod} m),\ l+1(\mathrm{mod} n)$. Then it is easily to check that $\theta,\ \tau$ and σ are automorphisms of the graph $\widehat{Q}_{m,n,k}$ and the subgroup $\langle \theta, \tau, \sigma \rangle$ acts transitively on $V(\widehat{Q}_{m,n,k})$.

Now we define a labeling $l_{\widehat{Q}}$ on vertices of $\widehat{Q}_{m,n,k}$ by

$$\begin{aligned} &l_{\widehat{Q}}(x_0^{(i)}) = l_{\widehat{Q}}(y_0^{(i)}) = \cdots = l_{\widehat{Q}}(z_0^{(i)}) = 1_\Gamma, \\ &l_{\widehat{Q}}(x_l^{(i)}) = l_{\widehat{Q}}(y_l^{(i)}) = \cdots = l_{\widehat{Q}}(z_l^{(i)}) = x_l, \quad 1 \le i \le m,\ 1 \le l \le n. \end{aligned}$$

Then we know that $l_G(E(G)) = \{1_\Gamma, x_1, x_2, \cdots, x_n\}$ and

$$l_G(E(G)) = \{1_\Gamma, x_1^2, x_1 \circ x_2, x_2^2, x_2 \circ x_3, \cdots, x_{n-1} \circ x_n, x_n^2\}. \qquad \square$$

Particularly, let Γ be a subgroup of $(\mathbb{Z}_{111111111}, \times)$ generated by

$$\{1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111\}$$

and $m = 1$. We get the symmetric sequence on a symmetric graph shown in Fig.3.2 again. Let $m = 5, n = 3$ and $k = 2$, i.e., the graph $\widehat{Q}_{5,3,2}$ with a labeling $l_G : V(\widehat{Q}_{5,3,2}) \to \{1_\Gamma, x_1, x_2, x_3, x_4\}$ is shown in Fig.4.1 following.

Denote by $N_G[x]$ all vertices in a graph G labeled by an element $x \in \Gamma$. Then we know the following result by Theorem 4.2. The following results are immediately conclusions by the proof of Theorem 4.3.

Corollary 4.3 *For integers $m, n \ge 1$, $\widehat{Q}_{m,n,k} \simeq C_m \times C_n \times C_k$.*

Corollary 4.4 $|N_{\widehat{Q}_{m,n,k}}[x]| = mk$ *for* $\forall x \in \{1_\Gamma, x_1, \cdots, x_n\}$ *and integers* $m, n, k \geq 1$.

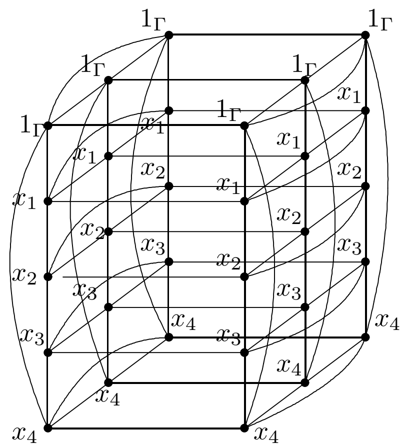

Fig.4.1

§5. Speculation

It should be noted that the essence we have done is a combinatorial notion, i.e., combining mathematical systems on that of graphs. Recently, Sridevi et al. consider the Fibonacci sequence on graphs in [16]. Let G be a graph and $\{F_0, F_1, F_2, \cdots, F_q, \cdots\}$ be the Fibonacci sequence, where F_q is the q^{th} Fibonacci number. An injective labeling $l_G : V(G) \to \{F_0, F_1, F_2, \cdots, F_q\}$ is called to be *super Fibonacci graceful* if the induced edge labeling by $l_G(u,v) = |l_G(u) - l_G(v)|$ is a bijection onto the set $\{F_1, F_2, \cdots, F_q\}$ with initial values $F_0 = F_1 = 1$. They proved a few graphs, such as those of $C_n \oplus P_m$, $C_n \oplus K_{1,m}$ have super Fibonacci labelings in [18]. For example, a super Fibonacci labeling of $C_6 \oplus P_6$ is shown in Fig.5.1.

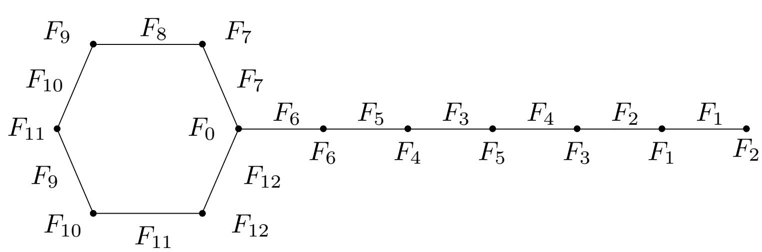

Fig.5.1

All of these are not just one mathematical system. In fact, they are applications of Smarandache multi-space and CC conjecture for developing modern mathematics, which appeals one to find combinatorial structures for classical mathematical systems, i.e., the following problem.

Problem 5.1 *Construct classical mathematical systems combinatorially and characterize them.*

For example, classical algebraic systems, such as those of groups, rings and fields by combinatorial principle.

Generally, a Smarandache multi-space is defined by the following.

Definition 5.2([6],[14]) *For an integer $m \geq 2$, let $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ be m mathematical systems different two by two. A Smarandache multi-space is a pair $(\widetilde{\Sigma};\widetilde{\mathcal{R}})$ with*

$$\widetilde{\Sigma} = \bigcup_{i=1}^{m} \Sigma_i, \quad and \quad \widetilde{\mathcal{R}} = \bigcup_{i=1}^{m} \mathcal{R}_i.$$

Definition 5.3([10]) *A combinatorial system $\mathscr{C}_G$ is a union of mathematical systems $(\Sigma_1;\mathcal{R}_1)$, $(\Sigma_2;\mathcal{R}_2)$, $\cdots$, $(\Sigma_m;\mathcal{R}_m)$ for an integer m, i.e.,*

$$\mathscr{C}_G = (\bigcup_{i=1}^{m} \Sigma_i; \bigcup_{i=1}^{m} \mathcal{R}_i)$$

with an underlying connected graph structure G, where

$$V(G) = \{\Sigma_1, \Sigma_2, \cdots, \Sigma_m\}, \quad E(G) = \{\ (\Sigma_i, \Sigma_j) \mid \Sigma_i \bigcap \Sigma_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

We have known a few Smarandache multi-spaces in classical mathematics. For examples, these rings and fields are group multi-space, and topological groups, topological rings and topological fields are typical multi-space are both groups, rings, or fields and topological spaces. Usually, if $m \geq 3$, a Smarandache multi-space must be underlying a combinatorial structure G. Whence, it becomes a combinatorial space in that case. I have presented the CC conjecture for developing modern mathematical science in 2005 [5], then formally reported it at *The 2th Conference on Graph Theory and Combinatorics of China* (2006, Tianjing, China)([7]-[10]).

CC Conjecture(Mao, 2005) *Any mathematical system $(\Sigma;\mathcal{R})$ is a combinatorial system $\mathscr{C}_G(l_{ij}, 1 \leq i, j \leq m)$.*

This conjecture is not just an open problem, but more likes a deeply thought, which opens a entirely way for advancing the modern mathematical sciences. In fact, it indeed means a *combinatorial notion* on mathematical objects following for researchers.

(1) There is a combinatorial structure and finite rules for a classical mathematical system, which means one can make combinatorialization for all classical mathematical subjects.

(2) One can generalizes a classical mathematical system by this combinatorial notion such that it is a particular case in this generalization.

(3) One can make one combination of different branches in mathematics and find new results after then.

(4) One can understand our WORLD by this combinatorial notion, establish combinatorial models for it and then find its behavior, for example,

what is true colors of the Universe, for instance its dimension?

and $\cdots$. For its application to geometry and physics, the reader is refereed to references [5]-[10], particularly, the book [10] of mine.

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Supermagic Coverings of Some Simple Graphs

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Abstract: A simple graph $G = (V, E)$ admits an H-covering if every edge in E belongs to a subgraph of G isomorphic to H. We say that G is Smarandachely pair $\{s, l\}$ H-magic if there is a total labeling $f : V \cup E \to \{1, 2, 3, \cdots, |V| + |E|\}$ such that there are subgraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ of G isomorphic to H, the sum $\sum\limits_{v\in V_1} f(v) + \sum\limits_{e\in E_1} f(e) = s$ and $\sum\limits_{v\in V_2} f(v) + \sum\limits_{e\in E_2} f(e) = l$. Particularly, if $s = l$, such a Smarandachely pair $\{s, l\}$ H-magic is called H-magic and if $f(V) = \{1, 2, \cdots, |V|\}$, G is said to be a H-supermagic. In this paper we show that edge amalgamation of a finite collection of graphs isomorphic to any 2-connected simple graph H is H-supermagic.

Key Words: H-covering, Smarandachely pair $\{s, l\}$ H-magic, H-magic, H-supermagic.

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§1. Introduction

The concept of H-magic graphs was introduced in [3]. An edge-covering of a graph G is a family of different subgraphs $H_1, H_2, \ldots, H_k$ such that each edge of E belongs to at least one of the subgraphs $H_i, 1 \le i \le k$. Then, it is said that G admits an $(H_1, H_2, \ldots, H_k)$ - edge covering. If every H_i is isomorphic to a given graph H, then we say that G admits an H-covering.

Suppose that $G = (V, E)$ admits an H-covering. We say that a bijective function $f : V \cup E \to \{1, 2, 3, \cdots, |V| + |E|\}$ is an H-magic labeling of G if there is a positive integer $m(f)$, which we call magic sum, such that for each subgraph $H' = (V', E')$ of G isomorphic to H, we have,$f(H') = \sum_{v\in V'} f(v) + \sum_{e\in E'} f(e) = m(f)$. In this case we say that the graph G is H-magic. When $f(V) = \{1, 2,\ |V|\}$, we say that G is H-supermagic and we denote its supermagic-sum by $s(f)$.

We use the following notations. For any two integers $n < m$, we denote by $[n, m]$, the set of all consecutive integers from n to m. For any set $\mathbb{I} \subset \mathbb{N}$ we write, $\sum \mathbb{I} = \sum\limits_{x\in\mathbb{I}} x$ and for any integers k, $\mathbb{I}+k = \{x+k : x \in \mathbb{I}\}$. Thus $k+[n, m]$ is the set of consecutive integers from $k+n$ to

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$k+m$. It can be easily verified that $\sum(\mathbb{I}+k) = \sum \mathbb{I}+k|\mathbb{I}|$. If $\mathbb{P} = \{X_1, X_2, \cdots, X_n\}$ is a partition of a set X of integers with the same cardinality then we say $\mathbb{P}$ is an n-equipartition of X. Also we denote the set of subsets sums of the parts of $\mathbb{P}$ by $\sum \mathbb{P} = \{\sum X1, \sum X_2, \cdots, \sum X_n\}$.Finally, given a graph $G = (V, E)$ and a total labeling f on it we denote by $f(G) = \sum f(V) + \sum f(E)$.

§2. Preliminary Results

In this section we give some lemmas which are used to prove the main results in Section 3.

Lemma 2.1 *Let h and k be two positive integers and h is odd. Then there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \cdots, X_k\}$ of $X = [1, hk]$ such that $\sum X_r = \dfrac{(h-1)(hk+k+1)}{2} + r$ for $1 \le r \le k$. Thus, $\sum \mathbb{P}$ is a set of consecutive integers given by $\sum \mathbb{P} = \dfrac{(h-1)(hk+k+1)}{2} + [1, k]$.*

Proof Let us arrange the set of integers $X = [1, hk]$ in a $h \times k$ matrix $\mathcal{A}$ as given below.

$$\mathcal{A} = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ n+1 & n+2 & \cdots & 2k-1 & 2k \\ 2n+1 & 2n+2 & \cdots & 3k-1 & 3k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (h-1)k+1 & (h-1)k+2 & \cdots & hk-1 & hk \end{pmatrix}_{h \times k}$$

That is, $\mathcal{A} = (a_{i,j})_{h \times k}$ where $a_{i,j} = (i-1)k + j$ for $1 \le i \le h$ and $1 \le j \le k$. For $1 \le r \le k$, define $X_r = \{a_{i,r}/1 \le i \le \frac{h+1}{2}\} \cup \{a_{i,k-r+1}/\frac{h+3}{2} \le i \le h\}$. Then

$$\begin{aligned} \sum X_r &= \sum_{i=1}^{\frac{h+1}{2}} a_{i,r} + \sum_{i=\frac{h+3}{2}}^{h} a_{i,k-r+1} \\ &= \sum_{i=1}^{\frac{h+1}{2}} (i-1)k + r \sum_{i=\frac{h+3}{2}}^{h} (i-1)k + k - r + 1 \\ &= \frac{h^2k + h - k - 1}{2} + r \\ &= \frac{(h-1)(hk+k+1)}{2} + r \quad \text{for} \quad 1 \le r \le k. \end{aligned}$$

Hence, $\sum \mathbb{P} = \dfrac{(h-1)(hk+k+1)}{2} + [1, k]$. □

Example 2.2 Let $h = 9$, $k = 6$ and $X = [1, 54]$. Then the partition subsets are $X_1 = \{1, 7, 13, 19, 25, 36, 42, 48, 54\}$, $X_2 = \{2, 8, 14, 20, 26, 35, 41, 47, 53\}$, $X_3 = \{3, 9, 15, 21, 27, 34, 40, 46, 52\}$, $X_4 = \{4, 10, 16, 22, 28, 33, 39, 45, 51\}$, $X_5 = \{5, 11, 17, 23, 29, 32, 38, 44, 50\}$ and $X_6 = \{6, 12, 18, 24, 30, 31, 37, 43, 49\}$. $\sum X_r = \dfrac{(h-1)(hk+k+1)}{2} + r = 244 + r$ for $1 \le r \le 6$.

Lemma 2.3 *Let h and k be two positive integers such that h is even and $k \geq 3$ is odd. Then there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \cdots, X_k\}$ of $X = [1, hk]$ such that $\sum X_r = \frac{(h-1)(hk+k+1)}{2} + r$ for $1 \leq r \leq k$. Thus,$\sum \mathbb{P}$ is a set of consecutive integers given by $\sum \mathbb{P} = \frac{(h-1)(hk+k+1)}{2} + [1, k]$.*

Proof Let us arrange the set of integers $X = \{1, 2, 3, \cdots, hk\}$in a $h \times k$ matrix $\mathcal{A}$ as given below.

$$\mathcal{A} = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ n+1 & n+2 & \cdots & 2k-1 & 2k \\ 2n+1 & 2n+2 & \cdots & 3k-1 & 3k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (h-1)k+1 & (h-1)k+2 & \cdots & hk-1 & hk \end{pmatrix}_{h \times k}$$

That is, $\mathcal{A} = (a_{i,j})_{h \times k}$ where $a_{i,j} = (i-1)k + j$ for $1 \leq i \leq h$ and $1 \leq j \leq k$. For $1 \leq r \leq k$, define $Y_r = \{a_{i,r}/1 \leq i \leq \frac{h}{2}\} \cup \{a_{i,k-r+1}/\frac{h}{2} + 1 \leq i \leq h-1\}$. Then

$$\begin{aligned}\sum Y_r &= \sum_{i=1}^{\frac{h}{2}} a_{i,r} + \sum_{i=\frac{h}{2}+1}^{h-1} a_{i,k-r+1} \\ &= \sum_{i=1}^{\frac{h}{2}} \{(i-1)k + r\} + \sum_{i=\frac{h}{2}+1}^{h-1} \{(i-1)k + k - r + 1\} \\ &= \frac{k(h-1)^2 + h - k - 2}{2} + r\end{aligned}$$

For $1 \leq r \leq k$, define $X_r = Y_{\sigma(r)} \cup \{(h-1)k + \pi(r)\}$, where σ and π denote the permutations of $\{1, 2, \cdots, k\}$ given by $\sigma(r) = \begin{cases} \frac{k-2r+1}{2} & \text{for} \quad 1 \leq r \leq \frac{k-1}{2} \\ \frac{3k-2r+1}{2} & \text{for} \quad \frac{k+1}{2} \leq r \leq k \end{cases}$ and $\pi(r) = \begin{cases} 2r & \text{for} \quad 1 \leq r \leq \frac{k-1}{2} \\ 2r-k & \text{for} \quad \frac{k+1}{2} \leq r \leq k \end{cases}$. Then

$$\begin{aligned}\sum X_r &= \sum Y_{\sigma(r)} + (h-1)k + \pi(r) \\ &= \frac{k(h-1)^2 + h - k - 2}{2} + \sigma(r) + (h-1)k + \pi(r)\end{aligned}$$

$\sum X_r = \begin{cases} \frac{k(h-1)^2 + h - k - 2}{2} + \frac{k-2r+1}{2} + (h-1)k + 2r & \text{for} \quad 1 \leq r \leq \frac{k-1}{2} \\ \frac{k(h-1)^2 + h - k - 2}{2} + \frac{3k-2r+1}{2} + (h-1)k + 2r - k & \text{for} \quad \frac{k+1}{2} \leq r \leq k \end{cases}$. On simplification we get $\sum X_r = \frac{(h-1)(hk+k+1)}{2} + r$ for $1 \leq r \leq k$. Hence, $\sum \mathbb{P} = \frac{(h-1)(hk+k+1)}{2} + [1, k]$. $\square$

Example 2.4 Let $h = 6$,$k = 5$ and $X = [1, 30]$. $Y_1 = \{1, 6, 11, 20, 25\}$, $Y_2 = \{2, 7, 12, 19, 24\}$, $Y_3 = \{3, 8, 13, 18, 23\}$, $Y_4 = \{4, 9, 14, 17, 22\}$and $Y_5 = \{5, 10, 15, 16, 21\}$. By definition the partition subsets are, $X_r = Y_{\sigma(r)} \cup \{(h-1)k + \pi(r)$ for $1 \leq r \leq 5$. $X_1 = \{2, 7, 12, 19, 24, 27\}$, $X_2 = \{1, 6, 11, 20, 25, 29\}$, $X_3 = \{5, 10, 15, 16, 21, 26\}$ $X_4 = \{4, 9, 14, 17, 22, 28\}$ $X_5 = \{3, 8, 13, 18, 23, 30\}$, Now, $\sum X_r = \dfrac{(h-1)(hk+k+1)}{2} + r = 90 + r$ for $1 \leq r \leq 5$.

Lemma 2.5 *If h is even, then there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \cdots, X_k\}$ of $X = [1, hk]$ such that $\sum X_r = \dfrac{h(hk+1)}{2}$ for $1 \leq r \leq k$. Thus, the subsets sum are equal and is equal to $\dfrac{h(hk+1)}{2}$.*

Proof Let us arrange the set of integers $X = \{1, 2, 3, \cdots, hk\}$in a $h \times k$ matrix $\mathcal{A}$ as given below.

$$\mathcal{A} = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ n+1 & n+2 & \cdots & 2k-1 & 2k \\ 2n+1 & 2n+2 & \cdots & 3k-1 & 3k \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (h-1)k+1 & (h-1)k+2 & \cdots & hk-1 & hk \end{pmatrix}_{h \times k}$$

That is, $\mathcal{A} = (a_{i,j})_{h \times k}$ where $a_{i,j} = (i-1)k + j$ for $1 \leq i \leq h$ and $1 \leq j \leq k$. For $1 \leq r \leq k$, define $X_r = \{a_{i,r}/1 \leq i \leq \frac{h}{2}\} \cup \{a_{i,k-r+1}/\frac{h}{2} + 1 \leq i \leq h-1\}$. Then

$$\begin{aligned} \sum X_r &= \sum_{i=1}^{\frac{h}{2}} a_{i,r} + \sum_{i=\frac{h}{2}+1}^{h} a_{i,k-r+1} \\ &= \sum_{i=1}^{\frac{h}{2}} \{(i-1)k + r\} + \sum_{i=\frac{h}{2}+1}^{h} \{(i-1)k + k - r + 1\} = \frac{h(hk+1)}{2} \end{aligned}$$

Thus, the subsets sum are equal and is equal to $\dfrac{h(hk+1)}{2}$. □

Example 2.6 Let $h = 6$, $k = 5$ and $X = [1, 30]$. Then the partition subsets are $X_1 = \{1, 6, 11, 20, 25, 30\}$, $X_2 = \{2, 7, 12, 19, 24, 29\}$, $X_3 = \{3, 8, 13, 18, 23, 28\}$, $X_4 = \{4, 9, 14, 17, 22, 27\}$ and $X_5 = \{5, 10, 15, 16, 21, 26\}$. Now, $\sum X_r = \dfrac{h(hk+1)}{2} = 93$ for $1 \leq r \leq 5$.

Lemma 2.7 *Let h and k be two even positive integers and $h \geq 4$. If $X = [1, hk+1] - \{\dfrac{k}{2} + 1\}$, there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \cdots, X_k\}$ of X such that $\sum X_r = \dfrac{h^2k + 3h - k - 2}{2} + r$ for $1 \leq r \leq k$. Thus $\sum \mathbb{P}$ is a set of consecutive integers $\dfrac{h^2k + 3h - k - 2}{2} + [1, k]$.*

Proof First we prove this lemma for $h = 2$ and we generalize for any even integer $h \geq 4$.

Case 1: $h = 2$.

$X = [1, 2k+1] - \{\frac{k}{2}+1\}$. For $1 \le r \le k$, define

$$X_r = \begin{cases} \{\frac{k}{2}+1-r, k+1+2r\} & \text{for} \quad 1 \le r \le \frac{k}{2} \\ \{\frac{3k}{2}+2-r, 2r\} & \text{for} \quad \frac{k}{2}+1 \le r \le k \end{cases}.$$

Hence, $\sum X_r = \frac{3k}{2}+2+r$ for $1 \le r \le k$.

Case 2: $h \ge 4$

Let $Y = [1, 2k+1] - \{\dfrac{k}{2}+1\}$ and $Z = [2k+2, hk+1]$. Then $X = Y \cup Z$. By Case 1, there exists a k-equipartition $\mathbb{P}_1 = \{Y_1, Y_2, \cdots, Y_k\}$ of Y such that

$$\sum Y_r = \frac{3k}{2}+2+r \quad \text{for} \quad 1 \le r \le k \tag{1}$$

Since $h-2$ is even, by Lemma 2.5, there exists a k-equipartition $\mathbb{P}'_2 = \{Z'_1, Z'_2, \cdots, Z'_k\}$ of $[1, (h-2)k]$ such that $\sum Z'_r = \dfrac{(h-2)(hk-2k+1)}{2}$ for $1 \le r \le k$. Adding $2k+1$ to $[1, (h-2)k]$, we get a k-equipartition $\mathbb{P}_2 = \{Z_1, Z_2, \cdots, Z_k\}$ of $Z = [2k+2, hk+1]$ such that $\sum Z_r = (h-2)(2k+1) + \dfrac{(h-2)(hk-2k+1)}{2}$ for $1 \le r \le k$. Let $X_r = Y_r \cup Z_r$ for $1 \le r \le k$. Then,

$$\begin{aligned} \sum X_r &= \sum Y_r \cup \sum Z_r \\ &= \frac{h^2k+3h-k-2}{2} + r \quad \text{for} \quad 1 \le r \le k. \end{aligned}$$

Hence, $\sum \mathbb{P}$ is a set of consecutive integers $\dfrac{h^2k+3h-k-2}{2} + [1, k]$. $\square$

Example 2.8 Let $h = 6$, $k = 6$ and $X = [1, 37] - \{4\}$.Then the partition subsets are $X_1 = \{3, 9, 14, 20, 31, 37\}$, $X_2 = \{2, 11, 15, 21, 30, 36\}$, $X_3 = \{1, 13, 16, 22, 29, 35\}$, $X_4 = \{7, 8, 17, 23, 28, 34\}$, $X_5 = \{6, 10, 18, 24, 27, 33\}$ and $X_6 = \{5, 12, 19, 25, 26, 32\}$. Now,

$$\sum X_r = \frac{h^2k+3h-k-2}{2} + r = 113 + r$$

for $1 \le r \le 6$.

Lemma 2.9 *Let h and k be two even positive integers. If $X = [1, hk+2] - \{1, \dfrac{k}{2}+2\}$, there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \cdots, X_k\}$ of X such that $\sum X_r = \dfrac{h^2k+5h-k-2}{2} + r$ for $1 \le r \le k$. Thus $\sum \mathbb{P}$ is a set of consecutive integers $\dfrac{h^2k+5h-k-2}{2} + [1, k]$.*

Proof First we prove this lemma for $h = 2$ and we generalize for any even integer $h \ge 4$.

Case 1: $h = 2$

$X = [1, 2k+2] - \{1, \dfrac{k}{2}+2\}$. For $1 \le r \le k$, define

$$X_r = \begin{cases} \{\dfrac{k}{2}+1-r, k+2+2r\} & \text{for} \quad 1 \le r \le \dfrac{k}{2}, \\ \{\dfrac{3k}{2}+3-r, 2r+1\} & \text{for} \quad \dfrac{k}{2}+1 \le r \le k. \end{cases}$$

Hence, $\sum X_r = \dfrac{3k}{2} + 4 + r$ for $1 \leq r \leq k$.

Case 2: $h \geq 4$

Let $Y = [1, 2k+2] - \{1, \dfrac{k}{2} + 2\}$ and $Z = [2k+3, hk+2]$. Then $X = Y \cup Z$. By Case 1, there exists a k-equipartition $\mathbb{P}_1 = \{Y_1, Y_2, \cdots, Y_k\}$ of Y such that

$$\sum Y_r = \frac{3k}{2} + 4 + r \quad \text{for} \quad 1 \leq r \leq k \tag{2}$$

Since $h-2$ is even, by Lemma 2.5, there exists a k-equipartition
$\mathbb{P}'_2 = \{Z'_1, Z'_2, \cdots, Z'_k\}$ of $[1, (h-2)k]$ such that $\sum Z'_r = \dfrac{(h-2)(hk-2k+1)}{2}$ for $1 \leq r \leq k$. Adding $2k+2$ to $[1, (h-2)k]$, we get a k-equipartition $\mathbb{P}_2 = \{Z_1, Z_2, \cdots, Z_k\}$ of $Z = [2k+3, hk+2]$ such that $\sum Z_r = (h-2)(2k+2) + \dfrac{(h-2)(hk-2k+1)}{2}$ for $1 \leq r \leq k$. Let $X_r = Y_r \cup Z_r$ for $1 \leq r \leq k$. Then,

$$\begin{aligned} \sum X_r &= \sum Y_r \cup \sum Z_r \\ &= \frac{h^2k + 5h - k - 2}{2} + r \quad \text{for} \quad 1 \leq r \leq k. \end{aligned}$$

Hence, $\sum \mathbb{P}$ is a set of consecutive integers $\dfrac{h^2k + 5h - k - 2}{2} + [1, k]$. □

Example 2.10 Let $h = 6$, $k = 6$ and $X = [1, 38] - \{1, 5\}$.Then the partition subsets are $X_1 = \{4, 10, 15, 21, 32, 38\}$, $X_2 = \{3, 12, 16, 22, 31, 37\}$, $X_3 = \{2, 14, 17, 23, 30, 36\}$, $X_4 = \{8, 9, 18, 24, 29, 35\}$, $X_5 = \{7, 11, 19, 25, 28, 34\}$ and $X_6 = \{6, 13, 20, 26, 27, 33\}$. Now, $\sum X_r = \dfrac{h^2k + 5h - k - 2}{2} + r = 119 + r$ for $1 \leq r \leq 6$.

§3. Main Results

Definition 3.1(Edge amalgamation of a finite collection of graphs, [1]) *For any finite collection (G_i, u_iv_i) of graphs G_i, each with a fixed edge u_iv_i, Carlson [1] defined the edge amalgamation $\mathcal{E}dgeamal\{(G_i, u_iv_i)\}$ as the graph obtained by taking the union of all the G_i's and identifying their fixed edges.*

Definition 3.2(Generalized Book) *If all the G_i's are cycles then $\mathcal{E}dgeamal\{(G_i, u_iv_i)\}$ is called a generalized book.*

Theorem 3.3 *Let H be a 2-connected (p, q) simple graph. Then the edge amalgamation $\mathcal{E}dgeamal\{(H_i, u_iv_i)\}$ of any finite collection $\{H_i, u_iv_i\}$ of graphs H_i, each with a fixed edge u_iv_i isomorphic to H is H-supermagic for all values of p and q.*

Proof Let $\{H_i, u_iv_i\}$ be a collection of n graphs H_i, each with a fixed edge u_iv_i and isomorphic to a 2-connected simple graph H.
Let $G = \mathcal{E}dgeamal\{(H_i, u_iv_i)\}$ with vertex set V and edge set E. Note that $|V| = n(p-2)+2$ and $|E| = n(q-1)+1$. Let $H_i = (V_i, E_i)$ for $1 \leq i \leq n$. Label the common edge of G as $e = w_1w_2$. Let $V'_i = V_i - \{w_1, w_2\}$ and $E'_i = E - \{e\}$ for $1 \leq i \leq n$.

Case 1: n is odd

Subcase (i): p is even and q is odd

Since $p-2$ and $q-1$ are even by Lemma 2.5 there exists n-equipartitions $\mathbb{P}_1' = \{X_1', X_2', \cdots, X_n'\}$ of $[1,(p-2)n]$ and $\mathbb{P}_2' = \{Y_1', Y_2', \cdots, Y_n'\}$ of $[1,(q-1)n]$ such that

$$\sum X_i' = \frac{(p-2)(pn-2n+1)}{2}, \quad \sum Y_i' = \frac{(q-1)(qn-n+1)}{2}.$$

Add 2 to each element of the set $[1,(p-2)n]$ and $(p-2)n+3$ to each element of the set $[1,(q-1)n]$. We get n-equipartitions $\mathbb{P}_1 = \{X_1, X_2, \cdots, X_n\}$ of $[3, pn-2n+3]$ and $\mathbb{P}_2 = \{Y_1, Y_2, \cdots, Y_n\}$ of $[pn-2n+4, (p+q-3)n+3]$ such that

$$\sum X_i = (p-2)2 + \frac{(p-2)(pn-2n+1)}{2}, \quad \sum Y_i = (q-1)(pn-2n+3) + \frac{(q-1)(qn-n+1)}{2}.$$

Define a total labeling $f : V \cup E \to [1,(p+q-3)n+3]$ as follows:

$$\begin{aligned} f(w_1) &= 1 \quad \text{and} \quad f(w_2) = 2. \\ f(e) &= pn-2n+3. \\ f(V_i') &= X_i \quad \text{for} \quad 1 \le i \le n. \\ f(E_i') &= Y_{n-i+1} \quad \text{for} \quad 1 \le i \le n. \end{aligned}$$

Then for $1 \le i \le n$,

$$\begin{aligned} f(H_i) &= f(w_1) + f(w_2) + f(e) + \sum f(V_i') + \sum f(E_i') \\ &= f(w_1) + f(w_2) + f(e) + \sum X_i' + \sum Y_{n-i+1}' \\ &= \frac{n(p+q)^2 + p + q + 5(n-1)}{2} - (n-1)(2p+3q) \\ &= \mathit{constant}. \end{aligned}$$

Since $H_i \cong H$ for $1 \le i \le n$, G is H-supermagic.

Subcase (ii): p is odd and q is even

Since $p-2$ and $q-1$ are odd, by Lemma 2.1 there exists n-equipartitions $\mathbb{P}_1' = \{X_1', X_2', \cdots, X_n'\}$ of $[1,(p-2)n]$ and $\mathbb{P}_2' = \{Y_1', Y_2', \cdots, Y_n'\}$ of $[1,(q-1)n]$ such that

$$\sum X_i' = \frac{(p-3)(pn-n+1)}{2} + i, \quad \sum Y_i' = \frac{(q-2)(qn+1)}{2} + i$$

for $1 \le i \le n$. Add 2 to each element of the set $[1,(p-2)n]$ and $(p-2)n+3$ to each element of the set $[1,(q-1)n]$. We get n-equipartitions $\mathbb{P}_1 = \{X_1, X_2, \cdots, X_n\}$ of $[3, pn-2n+3]$ and $\mathbb{P}_2 = \{Y_1, Y_2, \cdots, Y_n\}$ of $[pn-2n+4, (p+q-3)n+3]$ such that

$$\sum X_i = (p-2)2 + \frac{(p-3)(pn-n+1)}{2} + i, \quad \sum Y_i = (q-1)(pn-2n+3) + \frac{(q-2)(nq+1)}{2} + i$$

for $1 \le i \le n$. Define a total labeling $f : V \cup E \to [1,(p+q-3)n+3]$ as follows:

$$\begin{aligned} f(w_1) &= 1 \quad \text{and} \quad f(w_2) = 2. \\ f(e) &= pn-2n+3. \\ f(V_i') &= X_i \quad \text{for} \quad 1 \le i \le n. \\ f(E_i') &= Y_{n-i+1} \quad \text{for} \quad 1 \le i \le n. \end{aligned}$$

Then for $1 \leq i \leq n$,

$$\begin{aligned} f(H_i) &= f(w_1) + f(w_2) + f(e) + \sum f(V_i') + \sum f(E_i') \\ &= f(w_1) + f(w_2) + f(e) + \sum X_i' + \sum Y_{n-i+1}' \\ &= \frac{n(p+q)^2 + p + q + 5(n-1)}{2} - (n-1)(2p+3q) \\ &= constant. \end{aligned}$$

Since $H_i \cong H$ for $1 \leq i \leq n$, G is H-supermagic.

Subcase (iii): p and q are odd

Since $p-2$ is odd, by Lemma 2.1 there exists an n-equipartition $\mathbb{P}_1' = \{X_1', X_2', \cdots, X_n'\}$ of $[1,(p-2)n]$ such that $\sum X_i' = \dfrac{(p-3)(pn-n+1)}{2} + i$ for $1 \leq i \leq n$. Since $q-1$ is even and n is odd, by Lemma 2.3 there exists an n-equipartition $\mathbb{P}_2' = \{Y_1', Y_2', \cdots, Y_n'\}$ of $[1,(q-1)n]$ such that $\sum Y_i' = \dfrac{(q-2)(qn+1)}{2} + i$ for $1 \leq i \leq n$. Add 2 to each element of the set $[1,(p-2)n]$ and $(p-2)n+3$ to each element of the set $[1,(q-1)n]$. We get n-equipartitions $\mathbb{P}_1 = \{X_1, X_2, \cdots, X_n\}$ of $[3, pn-2n+3]$ and $\mathbb{P}_2 = \{Y_1, Y_2, \cdots, Y_n\}$ of $[pn-2n+4, (p+q-3)n+3]$ such that

$$\begin{aligned} \sum X_i &= (p-2)2 + \frac{(p-3)(np-n+1)}{2} + i, \\ \sum Y_i &= (q-1)(pn-2n+3) + \frac{(q-2)(qn+1)}{2} + i \end{aligned}$$

for $1 \leq i \leq n$. Define a total labeling $f : V \cup E \to [1,(p+q-3)n+3]$ as follows:

$$\begin{aligned} f(w_1) &= 1 \quad \text{and} \quad f(w_2) = 2. \\ f(e) &= pn-2n+3. \\ f(V_i') &= X_i \quad \text{for} \quad 1 \leq i \leq n. \\ f(E_i') &= Y_{n-i+1} \quad \text{for} \quad 1 \leq i \leq n. \end{aligned}$$

Then for $1 \leq i \leq n$,

$$\begin{aligned} f(H_i) &= f(w_1) + f(w_2) + f(e) + \sum f(V_i') + \sum f(E_i') \\ &= f(w_1) + f(w_2) + f(e) + \sum X_i' + \sum Y_{n-i+1}' \\ &= \frac{n(p+q)^2 + p + q + 5(n-1)}{2} - (n-1)(2p+3q) \\ &= constant. \end{aligned}$$

Since $H_i \cong H$ for $1 \leq i \leq n$, G is H-supermagic.

Subcase (iv): p and q are even

Since $p-2$ is even and n is odd, by Lemma 2.3 there exists an n-equipartition $\mathbb{P}_1' = \{X_1', X_2', \cdots, X_n'\}$ of $[1,(p-2)n]$ such that $\sum X_i' = \dfrac{(p-3)(pn-n+1)}{2} + i$ for $1 \leq i \leq n$. Since $q-1$ is odd, by Lemma 2.1 there exists an n-equipartition $\mathbb{P}_2' = \{Y_1', Y_2', \cdots, Y_n'\}$ of

$[1,(q-1)n]$ such that $\sum Y_i' = \dfrac{(q-2)(qn+1)}{2} + i$ for $1 \le i \le n$. Add 2 to each element of the set $[1,(p-2)n]$ and $(p-2)n+3$ to each element of the set $[1,(q-1)n]$. We get n-equipartitions $\mathbb{P}_1 = \{X_1, X_2, \cdots, X_n\}$ of $[3, pn-2n+3]$ and $\mathbb{P}_2 = \{Y_1, Y_2, \cdots, Y_n\}$ of $[pn-2n+4, (p+q-3)n+3]$ such that

$$\sum X_i = (p-2)2 + \frac{(p-3)(pn-n+1)}{2} + i, \quad \sum Y_i = (q-1)(pn-2n+3) + \frac{(q-2)(qn+1)}{2} + i$$

for $1 \le i \le n$. Define a total labeling $f : V \cup E \to [1,(p+q-3)n+3]$ as follows:

$$\begin{aligned}
f(w_1) &= 1 \quad \text{and} \quad f(w_2) = 2. \\
f(e) &= pn-2n+3. \\
f(V_i') &= X_i \quad \text{for} \quad 1 \le i \le n. \\
f(E_i') &= Y_{n-i+1} \quad \text{for} \quad 1 \le i \le n.
\end{aligned}$$

Then for $1 \le i \le n$,

$$\begin{aligned}
f(H_i) &= f(w_1) + f(w_2) + f(e) + \sum f(V_i') + \sum f(E_i') \\
&= f(w_1) + f(w_2) + f(e) + \sum X_i' + \sum Y_{n-i+1}' \\
&= \frac{n(p+q)^2 + p + q + 5(n-1)}{2} - (n-1)(2p+3q) \\
&= \mathit{constant}.
\end{aligned}$$

Since $H_i \cong H$ for $1 \le i \le n$, G is H-supermagic.

Case 2: n is even

Subcase (i): p is even and q is odd

The argument in Subcase(i) of Case (1) is independent of the nature of n. Hence we get G is H-supermagic.

Subcase (ii): p is odd and q is even

The argument in Subcase(ii) of Case (1) is independent of the nature of n. Hence we get G is H-supermagic.

Subcase (iii): p and q are odd

Since $p-2$ is odd, by Lemma 2.1 there exists an n-equipartition $\mathbb{P}_1' = \{X_1', X_2', \cdots, X_n'\}$ of $[1,(p-2)n]$ such that $\sum X_i' = \dfrac{(p-3)(pn-n+1)}{2} + i$ for $1 \le i \le n$. Since $q-1$ and n are even, by Lemma 2.7 there exists an n-equipartition $\mathbb{P}_2' = \{Y_1', Y_2', \cdots, Y_n'\}$ of $[1,(q-1)n+1] - \{\dfrac{n}{2}+1\}$ such that $\sum Y_i' = \dfrac{(q-1)^2 n + 3(q-1) - n - 2}{2} + i$ for $1 \le i \le n$. Add 2 to each element of the set $[1,(p-2)n]$ and $(p-2)n+2$ to each element of the set $[1,(q-1)n]$. We get n-equipartitions $\mathbb{P}_1 = \{X_1, X_2, \cdots, X_n\}$ of $[3, pn-2n+3]$ and $\mathbb{P}_2 = \{Y_1, Y_2, \cdots, Y_n\}$ of $[pn-2n+3, (p+q-3)n+3] - \{(p-2)n + \dfrac{n}{2} + 3\}$ such that

$$\sum X_i = (p-2)2 + \frac{(p-3)(pn-n+1)}{2} + i,$$

$$\sum Y_i = (q-1)(pn-2n+2) + \frac{(q-1)^2 n + 3(q-1) - n - 2}{2} + i$$

for $1 \leq i \leq n$. Define a total labeling $f : V \cup E \to [1, (p+q-3)n+3]$ as follows:

$$\begin{aligned} f(w_1) &= 1 \quad \text{and} \quad f(w_2) = 2. \\ f(e) &= (p-2)n + \frac{n}{2} + 3. \\ f(V_i') &= X_i \quad \text{for} \quad 1 \leq i \leq n. \\ f(E_i') &= Y_{n-i+1} \quad \text{for} \quad 1 \leq i \leq n. \end{aligned}$$

Then for $1 \leq i \leq n$,

$$\begin{aligned} f(H_i) &= f(w_1) + f(w_2) + f(e) + \sum f(V_i') + \sum f(E_i') \\ &= f(w_1) + f(w_2) + f(e) + \sum X_i' + \sum Y_{n-i+1}' \\ &= \frac{n(p+q)^2 + p + q}{2} - (n-1)(2p+3q-3) \\ &= constant. \end{aligned}$$

Since $H_i \cong H$ for $1 \leq i \leq n$, G is H-supermagic.

Subcase (iv): p and q are even

Since $p-2$ and n are even, by Lemma 2.9 there exists an n-equipartition $\mathbb{P}_1 = \{X_1, X_2, \cdots, X_n\}$ of $[1, (p-2)n+2] - \{1, \frac{n}{2}+2\}$ such that $\sum X_i = \frac{(p-2)^2 n + 5(p-2) - n - 2}{2} + i$ for $1 \leq i \leq n$.

Since $q-1$ is odd, by Lemma 2.1 there exists an n-equipartition $\mathbb{P}_2' = \{Y_1', Y_2', \cdots, Y_n'\}$ of $[1, (q-1)n]$ and $\sum Y_i' = \frac{(q-2)(qn+1)}{2} + i$ for $1 \leq i \leq n$. Add $(p-2)n+3$ to each element of the set $[1, (q-1)n]$. We get an n-equipartition $\mathbb{P}_2 = \{Y_1, Y_2, \cdots, Y_n\}$ of $[pn-2n+4, (p+q-3)n+3]$ such that $\sum Y_i = (q-1)(pn-2n+3) + \frac{(q-2)(qn+1)}{2} + i$ for $1 \leq i \leq n$. Define a total labeling $f : V \cup E \to [1, (p+q-3)n+3]$ as follows:

$$\begin{aligned} f(w_1) &= 1 \quad \text{and} \quad f(w_2) = \frac{n}{2} + 2. \\ f(e) &= pn - 2n + 3. \\ f(V_i') &= X_i \quad \text{for} \quad 1 \leq i \leq n. \\ f(E_i') &= Y_{n-i+1} \quad \text{for} \quad 1 \leq i \leq n. \end{aligned}$$

Then for $1 \leq i \leq n$,

$$\begin{aligned} f(H_i) &= f(w_1) + f(w_2) + f(e) + \sum f(V_i') + \sum f(E_i') \\ &= f(w_1) + f(w_2) + f(e) + \sum X_i' + \sum Y_{n-i+1}' \\ &= \frac{n(p+q)^2 + p + q}{2} - (n-1)(2p+3q-3) \\ &= constant. \end{aligned}$$

Since $H_i \cong H$ for $1 \leq i \leq n$, G is H-supermagic.

Hence, the edge amalgamation $\mathcal{E}dgeamal\{(H_i, u_iv_i)\}$ of any finite collection $\{H_i, u_iv_i\}$ of graphs H_i, each with a fixed edge u_iv_i and isomorphic to H is H-supermagic for all values of p and q. $\square$

Illustration 3.4 Let H_1,H_2,H_3,H_4 and H_5 be five graphs isomorphic to the wheel $W_4 = C_4 + K_1$ and their fixed edges given by dotted lines. Then the Edge amalgamation graph $\mathcal{E}dgeamal\{(H_i, u_iv_i)\}$ of the given collection is W_4-supermagic with supermagic sum 303.

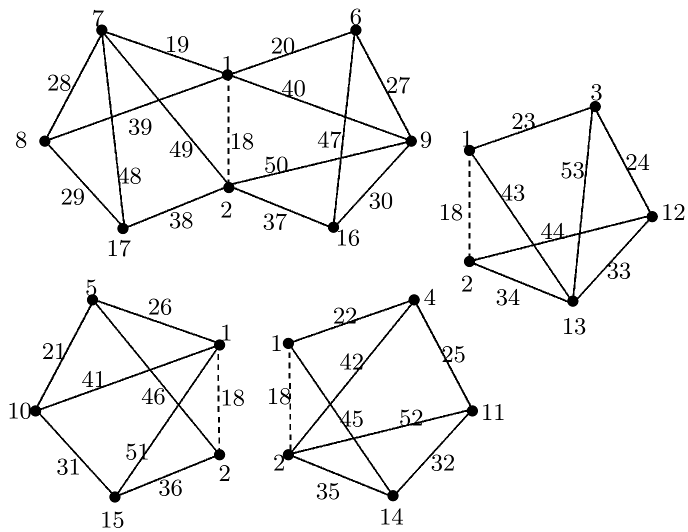

Fig.1

Illustration 3.5 Let H_1,H_2,H_3 and H_4 be four graphs isomorphic to H and their fixed edges given by dotted lines. Then the Edge amalgamation graph $\mathcal{E}dgeamal\{(H_i, u_iv_i)\}$ of the given collection is H-supermagic with supermagic sum 300.

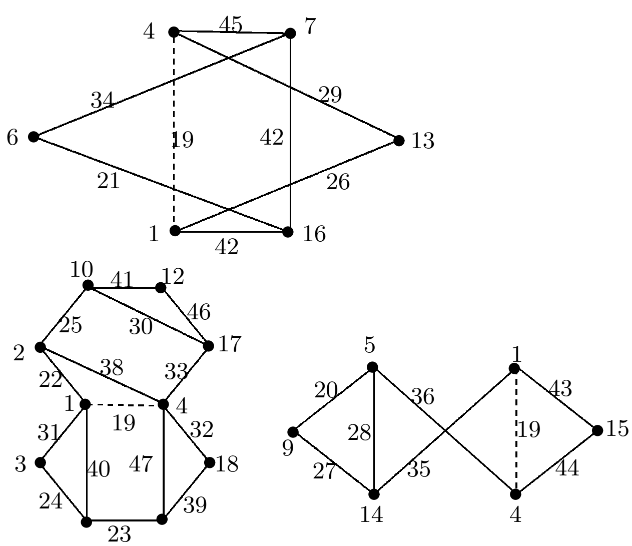

Fig.2

Definition 3.6(Book with m-gon pages) *Let n and m be any positive integers with $n \geq 1$ and $m \geq 3$. Then, n copies of the cycle C_m with an edge in common is called a book with n m-gon pages. That is, if $\{Gi, u_iv_i\}$ is a collection of n copies of the cycle C_m each with a fixed edge u_iv_i then $\mathcal{E}dgeamal\{(G_i, u_iv_i)\}$ is called a book with n m-gon pages.*

A book with 3 pentagon pages is given below.

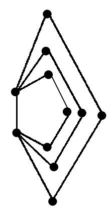

Fig.3

Corollary 3.7 *Books with n m-gon pages are C_m -supermagic for every positive integers $n \geq 1$and $m \geq 3$.*

Illustration 3.8 C_5-supermagic covering of a book with 3 hexagon pages is given below. The supermagic sum is 167.

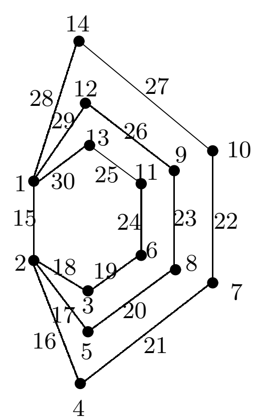

Fig.4

Theorem 3.9 *Let $H_i = K_{1,k}$ with vertex set $V(H_i) = \{v_i, v_{ir} : 1 \leq r \leq k\}$ and the edge set $E(H_i) = \{v_iv_{ir} : 1 \leq r \leq k\}$ where $1 \leq i \leq k$ and G be a graph obtained by joining a new vertex w with $v_{11}, v_{21}, \cdots, v_{k1}$. Then G is $K_{1,k}$-supermagic.*

Proof Let $V_i = \{v_i, v_{ir} : 1 \leq r \leq k\}$ and $E_i = \{V_iv_{jr} : 1 \leq r \leq k\}$ for $1 \leq i \leq k$. Then the vertex and edge set of $G = (V, E)$ are given by $V = \cup_{i=1}^{k} V_i \cup \{v\}$ and $E = \cup_{i=1}^{k} E_i \cup \{vv_1, vv_2, \cdots, vv_k\}$. Also $|V| = k^2 + k + 1$ and $|E| = k^2 + k$. Let $V_{k+1} = \{w, v_1, v_2, \cdots, v_k\}$ and $E_{k+1} = \{wv_1, wv_2, \cdots, wv_k\}$ and $H_{k+1} = (V_{k+1}, E_{k+1})$ be the graph with vertex set V_{k+1} and edge set E_{k+1}. Note that any edge of E belongs to at least one of the subgraphs H_i for $1 \leq i \leq k + 1$. Since $H_i \cong K_{1,k}$ for $1 \leq i \leq k + 1$, G admits a $K_{1,k}$-covering.

Case 1: k is odd

Since $k+1$ is even, by Lemma 2.3, there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \cdots, X_k\}$ of $X = [1, (k+1)k]$ such that

$$\sum X_i = \frac{k(k+1)^2}{2} + i \quad \text{for} \quad 1 \le i \le k \tag{3}$$

It can be easily verified from the definition of X_r in Lemma 2.3 that $\left(\frac{k+1}{2} - 1\right) k + \sigma(r) \in X_r$ for $1 \le r \le k$, where σ denotes the permutation of $\{1, 2, \cdots, k\}$ given by

$$\sigma(r) = \begin{cases} \dfrac{k-2r+1}{2} & \text{for} \quad 1 \le r \le \dfrac{k-1}{2} \\ \dfrac{3k-2r+1}{2} & \text{for} \quad \dfrac{k+1}{2} \le r \le k \end{cases}.$$

Construct $X_{k+1} = \{\left(\frac{k+1}{2} - 1\right) k + \sigma(r) : 1 \le r \le k\} \cup \{k^2 + k + 1\}$.

$$\begin{aligned} \sum X_{k+1} &= \sum_{r=1}^{k} \left[\left(\frac{k+1}{2} - 1\right) k + \sigma(r)\right] + k^2 + k + 1 \\ &= \frac{k^2(k-1)}{2} + \frac{k(k+1)}{2} + k^2 + k + 1 \\ &= \frac{k(k+1)^2}{2} + k + 1 \end{aligned} \tag{4}$$

From (1) and (2) we have

$$\sum X_i = \frac{k(k+1)^2}{2} + i \quad \text{for} \quad 1 \le i \le k+1 \tag{5}$$

As k is odd, by Lemma 1, there exists a $k+1$-equipartition $\mathbb{Q}' = \{Y_1', Y_2', \cdots, Y_{k+1}'\}$ of the set $Y = [1, k(k+1)]$ such that $\sum Y_i' = \frac{(k-1)[(k+1)^2+1]}{2} + i$ for $1 \le i \le k+1$.

Adding $k^2 + k + 1$ to $[1, k(k+1)]$, we get a $k+1$-equipartition $\mathbb{Q} = \{Y_1, Y_2, \cdots, Y_{k+1}\}$ of the set $Y = [k^2 + k + 2, 2k^2 + 2k + 1]$ such that

$$\sum Y_i = k(k^2 + k + 1) + \frac{(k-1)[(k+1)^2+1]}{2} + i \quad \text{for} \quad 1 \le i \le k+1 \tag{6}$$

Define a total labeling $f : V \cup E \to [1, 2k^2 + 2k + 1]$ as follows:

(i) $f(w) = k^2 + k + 1$.

(ii) $f(V_i) = X_i$ with $f(v_{i1}) = \left(\frac{k+1}{2} - 1\right) k + \sigma(r)$ for $1 \le i \le k+1$.

(iii) $f(E_i) = Y_{k+2-i}$ for $1 \le i \le k+1$.

Then for $1 \le i \le k+1$,

$$\begin{aligned} f(H_i) &= \sum f(V_i) + \sum f(E_i) = \sum X_i + \sum Y_{k+2-i} \\ &= \frac{4k^3 + 5k^2 + 5k + 2}{2}, \end{aligned}$$

which is a constant. Since $H_i \cong K_{1,k}$ for $1 \le i \le k+1$, G is $K_{1,k}$-supermagic.

Case 2: k is even

Since $k+1$ is odd, by Lemma 1,there exists a k-equipartition $\mathbb{P} = \{X_1, X_2, \cdots, X_k\}$ of $X = [1, (k+1)k]$ such that

$$\sum X_i = \frac{k(k+1)^2}{2} + i \quad \text{for} \quad 1 \le i \le k \tag{7}$$

It can be easily verified from the definition of X_r in Lemma 2.3 that $\left(\frac{k+2}{2} - 1\right) k + r \in X_r$ for $1 \le r \le \frac{k}{2}$, and $\left(\frac{k}{2} - 1\right) k + r \in X_r$ for $\frac{k}{2} + 1 \le r \le k$. Construct $X_{k+1} = \{\left(\frac{k+2}{2} - 1\right) k + r : 1 \le r \le \frac{k}{2}\} \cup \{\left(\frac{k}{2} - 1\right) k + r : \frac{k}{2} + 1 \le r \le k\} \cup \{k^2 + k + 1\}$.

$$\begin{aligned}
\sum X_{k+1} &= \sum_{r=1}^{\frac{k}{2}} \left[\left(\frac{k+2}{2} - 1\right) k + r\right] + \sum_{\frac{k}{2}+1}^{k} \left[\left(\frac{k}{2} - 1\right) k + r\right] + k^2 + k + 1 \\
&= \frac{k^2(k-1)}{2} + \frac{k(k+1)}{2} + k^2 + k + 1 \\
&= \frac{k(k+1)^2}{2} + k + 1
\end{aligned} \tag{8}$$

From (5) and (6) we have

$$\sum X_i = \frac{k(k+1)^2}{2} + i \quad \text{for} \quad 1 \le i \le k+1 \tag{9}$$

As k is even, by Lemma 2.3, there exists a $k+1$-equipartition $\mathbb{Q}' = \{Y_1', Y_2', \cdots, Y_{k+1}'\}$ of the set $Y = [1, k(k+1)]$ such that $\sum Y_i' = \frac{(k-1)[(k+1)^2+1]}{2} + i$ for $1 \le i \le k+1$. Adding $k^2 + k + 1$ to $[1, k(k+1)]$, we get a $k+1$-equipartition $\mathbb{Q} = \{Y_1, Y_2, \cdots, Y_{k+1}\}$ of the set $Y = [k^2 + k + 2, 2k^2 + 2k + 1]$ such that

$$\sum Y_i = k(k^2 + k + 1) + \frac{(k-1)[(k+1)^2 + 1]}{2} + i \quad \text{for} \quad 1 \le i \le k+1 \tag{10}$$

Define a total labeling $f : V \cup E \to [1, 2k^2 + 2k + 1]$ as follows:

(i) $f(w) = k^2 + k + 1$.

(ii) $f(V_i) = X_i$ with $f(v_{i1}) = \left(\frac{k+2}{2} - 1\right) k + r$ for $1 \le i \le \frac{k}{2}$ and $f(v_{i1}) = \left(\frac{k}{2} - 1\right) k + r$ for $\frac{k}{2} + 1 \le i \le k$.

(iii) $f(E_i) = Y_{k+2-i}$ for $1 \le i \le k+1$.

Then for $1 \le i \le k+1$,

$$\begin{aligned}
f(H_i) &= \sum f(V_i) + \sum f(E_i) \\
&= \sum X_i + \sum Y_{k+2-i} \\
&= \frac{4k^3 + 5k^2 + 5k + 2}{2},
\end{aligned}$$

which is a constant. Since $H_i \cong K_{1,k}$ for $1 \leq i \leq k+1$, G is $K_{1,k}$-supermagic. Thus, in both the cases G is $K_{1,k}$-supermagic with supermagic sum $s(f) = \dfrac{4k^3+5k^2+5k+2}{2}$. $\square$

Illustration 3.10

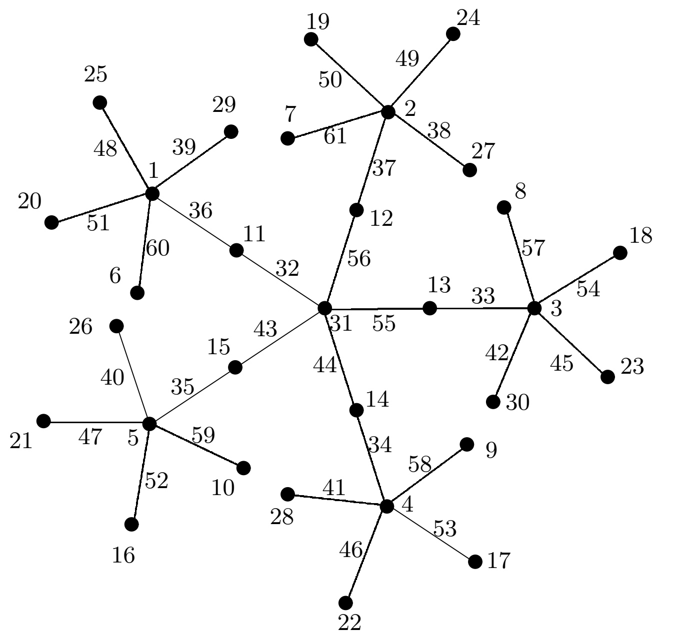

Fig.1. G**- is** $K_{1,5}$**-supermagic with supermagic sum 236.**

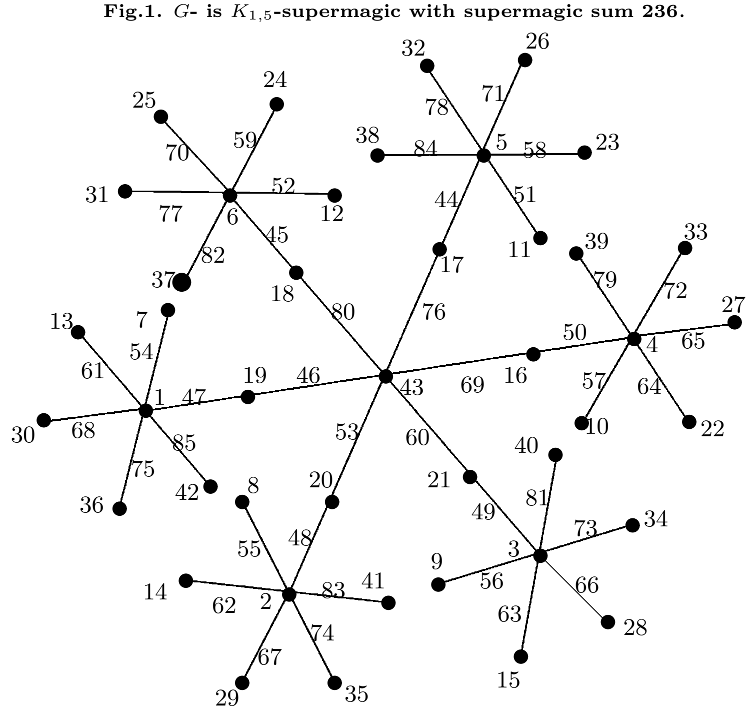

Fig.2. G**- is** $K_{1,6}$**-supermagic with supermagic sum 538.**

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Elementary Abelian Regular Coverings of Cube

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Abstract: For a give finite connected graph Γ, a group H of automorphisms of Γ and a finite group A, a natural question can be raised as follows: Find all the connected regular coverings of Γ having A as its covering transformation group, on which each automorphism in H can be lifted. In this paper, we classify all the connected regular covering graphs of the cube satisfying the following two properties: (1) the covering transformation group is isomorphic to the elementary Abelian p-groups; (2) the group of fibre-preserving automorphisms acts edge-transitively.

Key Words: Connected graph, graph covering, cube, Smarandachely covering, regular covering.

AMS(2010):

§1. Introduction

All graphs considered in this paper are finite, undirected and simple. For a graph Γ, we use $V(\Gamma)$, $E(\Gamma)$, $A(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its vertex set, edge set, arc set and full automorphism group, respectively. For any $v \in V(\Gamma)$, by $N(v)$ we denote the neighborhood of v in Γ. For an arc $(u, v) \in A(\Gamma)$, we denote the corresponding undirected edge by uv.

A graph $\widetilde{\Gamma}$ is called a *covering* of the graph Γ with projection $p : \widetilde{\Gamma} \to \Gamma$ if there is a surjection $p : V(\widetilde{\Gamma}) \to V(\Gamma)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(\Gamma)$ and $\tilde{v} \in p^{-1}(v)$. The graph $\widetilde{\Gamma}$ is called the *covering graph* and Γ is the *base graph*. A $p : \widetilde{\Gamma} \to \Gamma$ is called to be a *Smarandachely covering* of Γ if there exist $u, v \in V(\Gamma)$ such that $|p^{-1}(u)| \neq |p^{-1}(v)|$. Conversely, if $|p^{-1}(v)| = n$ for each $v \in V(\Gamma)$, then such a covering p is said to be *n-fold*. Each $p^{-1}(v)$ is called a *fibre* of $\widetilde{\Gamma}$. An automorphism of $\widetilde{\Gamma}$ which maps a fibre to a fibre is said to be *fibre-preserving*. The group K of all automorphisms of $\widetilde{\Gamma}$ which fix each of the fibres setwise is called the *covering transformation group*. A covering $p : \widetilde{\Gamma} \to \Gamma$ is said to be *regular* (simply, *A-covering*) if there is a subgroup A of K acting regularly on each fibre. Moreover, if Γ is connected, then $A = K$.

A purely combinatorial description of a covering was introduced through a voltage graph

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by Gross and Tucker [4,5] and also a very similar idea was appeared in Biggs' monograph [1,2]. Let A be a finite group. An (*ordinary*) *voltage assignment* (or, A-*voltage assignment*) of Γ is a function $\phi : A(\Gamma) \to A$ with the property that $\phi(u,v) = \phi(v,u)^{-1}$ for each $(u,v) \in A(\Gamma)$. The values of ϕ are called *voltages*, and A is called the *voltage group*. The *graph* $\Gamma \times_\phi A$ *derived from* ϕ is defined by $V(\Gamma \times_\phi A) = V(\Gamma) \times A$ and $E(\Gamma \times_\phi A) = \{((u,g),(v,\phi(u,v)g)) \mid (u,v) \in E(\Gamma),\ g \in A\}$. Clearly, the graph $\Gamma \times_\phi A$ is a covering of the graph Γ with the first coordinate projection $p : \Gamma \times_\phi A \to \Gamma$, which is called the *natural projection*. For each $u \in V(\Gamma)$, $\{(u,g) \mid g \in A\}$ is a fibre of $\Gamma \times_\phi A$. Moreover, by defining $(u,g')^g := (u, g'g)$ for any $g \in A$ and $(u,g') \in V(\Gamma \times_\phi A)$, A can be identified with a fibre-preserving automorphism subgroup of $\mathrm{Aut}(\Gamma \times_\phi A)$ acting regularly on each fibre. Therefore, p can be viewed as a A-*covering*. Given a spanning tree T of the graph Γ, a voltage assignment ϕ is called T-*reduced* if the voltages on the tree arcs are identity. Gross and Tucker ([4]) showed that every regular covering of a graph Γ can be derived from an ordinary T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of Γ.

An automorphism α of Γ can be lifted to an automorphism $\tilde{\alpha}$ of a covering graph $\widetilde{\Gamma}$ if $p\tilde{\alpha} = \alpha p$, where p is the covering projection from $\widetilde{\Gamma}$ to Γ. We say a subgroup of H of $\mathrm{Aut}(\Gamma)$ can be lifted if each element of H can be lifted.

For a given finite connected graph Γ, a group H of automorphisms of Γ and a finite group A, a natural question can be raised as follows: Find all the connected regular coverings of Γ having A as its covering transformation group, on which each automorphism in H can be lifted. In [3], Du, Kawk and Xu investigate the regular coverings with $A = Z_p^n$, an elementary Abelian group and get some new matrix-theoretical characterizations for an automorphism of the base graph to be lifted, and as one of the applications, they gave a classification of all connected regular coverings of the Petersen graph with the covering transformation group Z_p^n, whose fibre-preserving automorphism subgroup acts arc-transitively.

In this paper, we use the same method to classify all the connected regular covering graphs of the cube satisfying the following two properties: (1) the covering transformation group is isomorphic to the elementary Abelian p-group; (2) the group of fibre-preserving automorphisms acts edge-transitively.

The cube is identified with the graph Γ as shown in Figure (a). Fix a spanning tree T in Γ as shown in Figure (b). Let $V_1 = \{2,6,4,7,3,5\}$. Then the induced subgraph $\Gamma(V_1)$ is a line as shown in Figure (c).

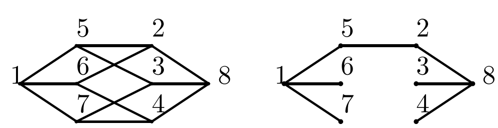

Figure (a): the graph Γ; (b) a spanning tree T of Γ.

Figure (c): the induced subgraph $\Gamma(V_1)$

First, we introduce five families of covering graphs $\Gamma \times_\phi Z_p^n$ of cube Γ by giving a T-reduced voltage assignment ϕ. Since ϕ is T-reduced, we only need to give the voltages on the cotree arcs (see Figure (c)). Let t denote the transposition of a matrix.

(1) $X(2,1) := \Gamma \times_\phi Z_2$, where $\phi_{26} = \phi_{47} = \phi_{35} = 1$ and $\phi_{46} = \phi_{37} = 0$,
$X(p,1) := \Gamma \times_\phi Z_p$, where $p = 3$ or $p \equiv 1 (\mathrm{mod}\ \ 6)$, $\phi_{26} = \phi_{37} = 1$, $\phi_{46} = \phi_{35} = \frac{1+\sqrt{-3}}{2}$ and $\phi_{47} = 0$.

(2) $X(p,2) := \Gamma \times_\phi Z_p^2$, where $\phi_{26} = \phi_{37} = (0,1)$, $\phi_{46} = \phi_{35} = (1,0)$ and $\phi_{47} = (0,0)$.

(3) $X(p,3) := \Gamma \times_\phi Z_p^3$, where $((\phi_{26})^t, (\phi_{47})^t, (\phi_{35})^t) = I_{3\times 3}$, $\phi_{46} = (0,1,-1)$ and $\phi_{37} = (-1,1,0)$.

(4) $X(p,4) := \Gamma \times_\phi Z_p^4$, where $p = 3$ or $p \equiv 1 (\mathrm{mod}\ \ 6)$, $((\phi_{26})^t,\ (\phi_{46})^t,\ (\phi_{47})^t,\ (\phi_{37})^t) = I_{4\times 4}$ and $\phi_{35} = (\dfrac{1-\sqrt{-3}}{2},\ -1,\ \dfrac{1+\sqrt{-3}}{2},\ \dfrac{1-\sqrt{-3}}{2})$.

(5) $X(p,5) := \Gamma \times_\phi Z_p^5$, where $((\phi_{26})^t, (\phi_{46})^t, (\phi_{47})^t, (\phi_{37})^t, (\phi_{35})^t) = I_{5\times 5}$.

Now we state the main theorem of this paper.

Theorem 1.1 *Let $\widetilde{\Gamma}$ be a connected regular covering of the cube Γ whose covering transformation group is isomorphic to Z_p^n and whose fibre-preserving automorphism subgroup G acts edge-transitively on $\widetilde{\Gamma}$. Then, $\widetilde{\Gamma}$ is isomorphic to one of the graphs in* (1)-(5) *listed above. Moreover, for the graphs $X(2,1)$, $X(3,1)$, $X(p,2)$, $X(p,3)$, $X(3,4)$ and $X(p,5)$,* $\mathrm{Aut}(\Gamma)$ *can be lifted, and so they are 2-arc-transitive; and for the graphs $X(p,1)$ and $X(p,4)$ for $p \equiv 1 (\mathrm{mod}\ \ 6)$, the subgroup isomorphic to $A_4 \times Z_2$ can be lifted but* $\mathrm{Aut}\Gamma$ *cannot, and so they are arc-transitive, in particular, all these five families of graphs are vertex transitive.*

§2. Algorithm for the Lifting

In this section, we present the algorithm given by Du, Kwak and Xu [3], which deals with the lifting problem for regular coverings of a graph Γ whose covering transformation group is elementary Abelian.

Throughout this section, let Γ be a connected graph and let $\widetilde{\Gamma} = \Gamma \times_\phi Z_p^n$ be a connected regular covering of the Γ. The voltage group Z_p^n will be identified with the additive group of the n-dimensional vector space $V(n,p)$ over the finite field $GF(p)$. Since Γ is connected, the number $\beta(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1$ is equal to the number of independent cycles in Γ and it is referred to as the *Betti number* of Γ.

Let $V(\Gamma) = \{0, 1, \ldots, |V(\Gamma)| - 1\}$. For any arc $(i,j) \in A(\Gamma)$, by $\phi_{i,j}$ we denote the voltage on the arc, which is identified with a row vector in $V(n,p)$. An arc $(i,j) \in A(\Gamma)$ is called *positive* (resp. *negative*) if $i < j$ (resp. $i > j$). For each subset F in $E(\Gamma)$, we denote the set of its arcs, positive arcs and negative arcs by $A(F)$, $A^+(F)$ and $A^-(F)$, respectively, so that $A(F) = A^+(F) \cup A^-(F)$. In particular, if $F = E(\Gamma)$, we prefer to use $A(\Gamma)$, $A^+(\Gamma)$ and $A^-(\Gamma)$ to denote $A(F)$, $A^+(F)$ and $A^-(F)$, respectively. Fix a spanning tree T in the graph

Γ and let $E_0 = E(T)$, so that $|E(\Gamma) \setminus E_0|$ is the Betti number $\beta(\Gamma)$ of the graph Γ. From now on, the voltage assignment ϕ is assumed to be T-reduced. By the connectedness of $\widetilde{\Gamma}$, $\{\phi_{i,j} \mid (i,j) \in A^+(E(\Gamma) \setminus E_0)\}$ generates the group Z_p^n. Hence, we get $n \leqslant \beta(\Gamma)$.

Let E_1 be a set of edges such that $\phi_{A^+(E_1)} = \{\phi_{i,j} \bigm| (i,j) \in A^+(E_1)\}$ is a basis for the vector space $V(n,p)$, and let $E_2 = E(\Gamma) \setminus (E_0 \cup E_1)$. Let

$$|E_0| = k, \quad |E_1| = n \quad \text{and} \quad |E_2| = m, \tag{1}$$

so that the number of edges in Γ is $k+n+m$.

Let Φ_0 (resp. Φ_1 and Φ_2) be the $k \times n$ (resp. $n \times n$ and $m \times n$) matrix with the row vectors $\phi_{i,j}$ for the arcs (i,j) in $A^+(E_0)$ (resp. $A^+(E_1)$ and $A^+(E_2)$), according to a fixed order of the positive arcs. Since the row vectors of Φ_1 form a basis for $V(n,p)$, there exists an $m \times n$ matrix M, called a *voltage generating matrix* of ϕ, such that

$$\Phi_2 = M\Phi_1. \tag{2}$$

Let

$$\Phi = \begin{pmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \end{pmatrix}, \tag{3}$$

which is a $(k+n+m) \times n$ matrix over $GF(p)$, called a *voltage* (*assignment*) matrix corresponding to the voltage assignment ϕ. If we take $\phi_{A^+(E_1)}$ so that $\Phi_0 = \mathbf{0}$ and $\Phi_1 = I_{n\times n}$, the $n\times n$ identity matrix, then Φ is called a *reduced form* or a *T-reduced form* of the voltage assignment matrix Φ. From now on one may assume that Φ is in a reduced form without loss of any generality.

Let $\mathbf{V} = V(k+n+m,p)$ be the $(k+n+m)$-dimensional row vector space over the field $GF(p)$. Hereafter, we denote a vector $\mathbf{v}$ in $\mathbf{V}$ by $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2) \in Z_p^k \oplus Z_p^n \oplus Z_p^m$, where the coordinates of the vector $\mathbf{v}_0$ (resp. $\mathbf{v}_1$ and $\mathbf{v}_2$) are indexed by arcs in $A^+(E_0)$ (resp. $A^+(E_1)$ and $A^+(E_2)$), according to the same order of row vectors in Φ_0 (Φ_1 and Φ_2).

Given a graph Γ, its spanning tree T, and a positive cotree arc (u,v), there is a unique path from v to u in T which is denoted by $[v,\cdots,u]$. We call the closed walk $(u,[v,\cdots,u])$ the *fundamental cycle* belonging to (u,v), and denote it by $C(u,v;T)$. There are $n+m$ fundamental cycles in Γ, where $n+m$ is the Betti number of Γ.

Given a graph Γ and its spanning tree T, we keep the same order for positive arcs as the order of row vectors of the voltage matrix Φ. For each positive cotree arc (u,v), let $\mathbf{p}^{u,v}$ be the k-dimensional row vector over $GF(p)$ whose (i,j)-coordinate $\mathbf{p}^{u,v}_{i,j}$ indexed by the positive tree arc (i,j) of the given order is defined as follows:

$$\mathbf{p}^{u,v}_{i,j} = \begin{cases} 1 & \text{if } (i,j) \text{ is in } C(u,v;T), \\ -1 & \text{if } (j,i) \text{ is in } C(u,v;T), \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Let P be the $(n+m) \times k$ matrix whose row vectors are $\mathbf{p}^{u,v}$, indexed by the positive cotree arcs (u,v) of the given order. We call P the *incidence matrix* for the fundamental cycles of the graph Γ with respect to the tree T.

Now we state the algorithm for solving lifting problem for the connected regular coverings of a graph Γ whose covering transformation group is elementary Abelian.

(1st) Choose a fixed spanning tree T in Γ and write down the arcs in $A^+(E_0)$, $A^+(E_1)$ and $A^+(E_2)$ in a certain order so that $\Phi_0 = \mathbf{0}$, $\Phi_1 = I_{n\times n}$ and $\Phi_2 = M$.

(2nd) Calculate the incidence matrix P for the fundamental cycles of Γ with respect to T.

(3rd) Assume that the voltage generating matrix $M = (a_{ij})_{m\times n}$, where the entries a_{ij} are unknowns. Let $\Delta = ((-M, I_{m\times m})P, -M, I_{m\times m})$, whose columns are indexed by the arcs in $A^+(E_0), A^+(E_1), A^+(E_2)$ according to the given order. We call the matrix Δ the *discriminant matrix* for a lift of ϕ. For convenience, we write $\Delta_0 = (-M, I_{m\times m})P$, $\Delta_1 = -M$ and $\Delta_2 = I_{m\times m}$, so that $\Delta = (\Delta_0, \Delta_1, \Delta_2)$, as a block matrix.

(4th) Let $\Delta = (\cdots, \mathbf{c}_{i,j}, \cdots)$, where $c_{i,j}$ is the column indexed by $(i,j) \in A^+(\Gamma)$. For a given $\sigma \in \mathrm{Aut}(\Gamma)$, let $\mathbf{c}^\sigma_{i,j} = \mathbf{c}_{i^{\sigma^{-1}}, j^{\sigma^{-1}}}$, where we assume that $\mathbf{c}_{i,j} = -\mathbf{c}_{j,i}$ for any arc (i,j). Let $\Delta^\sigma = (\cdots, \mathbf{c}^\sigma_{i,j}, \cdots)$ for any $(i,j) \in A^+(\Gamma)$, and let $(\Delta^\sigma)_0$, $(\Delta^\sigma)_1$ and $(\Delta^\sigma)_2$ denote the first, the second and the third blocks of the matrix Δ^σ respectively, as before. Then one can say that

$$\sigma \text{ can be lifted} \iff (\Delta^\sigma)_1 + (\Delta^\sigma)_2 M = \mathbf{0} \iff \Delta^\sigma_1 + \Delta^\sigma_2 M = \mathbf{0}. \qquad (5)$$

Proof of Theorem 1.1

Let $E_0 = E(T)$ and $E = E(\Gamma) \setminus E_0$. Give an ordering for the arcs in $A^+(E_0)$ and $A^+(E)$ as follows:

$$A^+(E_0) = \{15, 16, 17, 25, 28, 38, 48\},$$
$$A^+(E) = \{26, 46, 47, 37, 35\}.$$

Give fundamental cycles in Γ as follows:

$$\begin{aligned}
C(2,6;T) &= (2,6,1,5,2),\\
C(4,6;T) &= (4,6,1,5,2,8,4),\\
C(4,7;T) &= (4,7,1,5,2,8,4),\\
C(3,7;T) &= (3,7,1,5,2,8,3),\\
C(3,5;T) &= (3,5,2,8,3).
\end{aligned}$$

It is well-known that $\mathrm{Aut}(\Gamma) \cong S_4 \times Z_2$. Take four automorphisms of Γ as follows: $\alpha = (243)(567)$, $\beta = (14)(23)(58)(67)$, $\gamma = (18)(27)(36)(45)$ and $\delta = (23)(67)$.

It is easy to check that $M =< \alpha, \beta >$ is subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to A_4, $N =< M, \gamma >$ is subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to $A_4 \times Z_2$, and $< N, \delta >= \mathrm{Aut}(\Gamma)$. Thus we have:

(1) M can be lifted if and only if α and β can be lifted;
(2) N can be lifted if and only if α, β and γ can be lifted;
(3) $\mathrm{Aut}(\Gamma)$ can be lifted if and only if α, β, γ and δ can be lifted.

Since $\beta(\Gamma)=5$, we get $n\leqslant 5$. If $n=5$, then Γ is nothing but $X(p,5)$ and by [6, Theorem 2.11], $\mathrm{Aut}(\Gamma)$ can be lifted. So, in what follows, we assume $n<5$ and divide our proof into four cases for each n with $1\leqslant n\leqslant 4$.

3.1 The Case of $n=1$

Suppose that $n=1$. Then $K=Z_p=V(1,p)$. Since the element $(18)(25)(36)(47)$ of $\mathrm{Aut}(\Gamma)$ maps 2, 6, 4, 7, 3, 5 to 5, 3, 7, 4, 6, 2, respectively, without loss of any generality, we have the following three essentially different cases for the set E_1 and E_2:

(1) $E_1=\{26\}$ and $E_2=\{46,47,37,35\}$;

(2) $E_1=\{46\}$ and $E_2=\{26,47,37,35\}$;

(3) $E_1=\{47\}$ and $E_2=\{26,46,37,35\}$.

Case (1): $E_1=\{26\}$ and $E_2=\{46,47,37,35\}$.

In this case, the incidence matrix is:

$$P=\begin{array}{c}\\ (2,6)\\ (4,6)\\ (4,7)\\ (3,7)\\ (3,5)\end{array}\begin{array}{c}\begin{array}{ccccccc}(1,5) & (1,6) & (1,7) & (2,5) & (2,8) & (3,8) & (4,8)\end{array}\\ \begin{pmatrix} 1 & -1 & 0 & -1 & 0 & 0 & 0\\ 1 & -1 & 0 & -1 & 1 & 0 & -1\\ 1 & 0 & -1 & -1 & 1 & 0 & -1\\ 1 & 0 & -1 & -1 & 1 & -1 & 0\\ 0 & 0 & 0 & -1 & 1 & -1 & 0\end{pmatrix}\end{array}.$$

Let $D=(D_0,D_1,D_2)$ be the discriminant matrix with $D_0=(-M,I_{4\times4})P$, $D_1=(-M)$ and $D_2=(I_{4\times4})$ with a voltage generating matrix $M=(a,b,c,d)^t$. A direct computation gives that $D_0=(-M,I_{4\times4})P$ is equal to

$$D_0=\begin{array}{c}\begin{array}{ccccccc}(1,5) & (1,6) & (1,7) & (2,5) & (2,8) & (3,8) & (4,8)\end{array}\\ \begin{pmatrix} -a+1 & a-1 & 0 & a-1 & 1 & 0 & -1\\ -b+1 & b & -1 & b-1 & 1 & 0 & -1\\ -c+1 & c & -1 & c-1 & 1 & -1 & 0\\ -d & d & 0 & d-1 & 1 & -1 & 0\end{pmatrix}\end{array}.$$

For an automorphism σ of Γ can be lifted if and only if

$$\Delta_1^\sigma+\Delta_2^\sigma M=(c_{26})^\sigma+(c_{46}c_{47}c_{37}c_{35})^\sigma M=\mathbf{0}. \tag{6}$$

Inserting $\alpha=(243)(567)$ to (6), we have

$$\begin{aligned}\mathbf{0} &= D_1^\alpha+D_2^\alpha M=(c_{26})^\alpha+(c_{46}c_{47}c_{37}c_{35})^\alpha M\\ &= (c_{35})+(c_{25}c_{26}c_{46}c_{47})M\\ &= \begin{pmatrix} a^2-a-ab+c\\ ab-a-b^2+d\\ ac-a-bc\\ ad-a-bd+1\end{pmatrix}=(x_{i,j}).\end{aligned}$$

Inserting $\beta = (14)(23)(58)(67)$ to (6), we have

$$\begin{aligned} \mathbf{0} &= D_1^\beta + D_2^\beta M = (c_{26})^\beta + (c_{46}c_{47}c_{37}c_{35})^\beta M \\ &= (c_{37}) + (c_{17}c_{16}c_{26}c_{28})M \\ &= \begin{pmatrix} ab - b - ac + d \\ -a + b^2 - bc + d \\ -a + bc - c^2 + d + 1 \\ bd - cd + d \end{pmatrix} = (y_{ij}). \end{aligned}$$

Now assume that α and β can be lifted. By $y_{41} = 0$, we distinguish two cases: (1) $d = 0$; (2) $b - c + 1 = 0$. If (1) happens, by $x_{41} = 0$ and $x_{11} = 0$, we have $a = 1$ and $b = c$. But it doesn't satisfy $y_{21} = 0$. If (2) happens, by $y_{11} = 0$, we have $-a - b + d = 0$. By $x_{21} = 0$, we distinguish two subcases: (i) $a - b + 1 = 0$; (ii) $b = 0$. If (i) happens, by $(x_{ij}) = 0$, we have that either $a = c = 0$, $b = d = 1$ and $p = 2$ or $a = d = 2$, $b = 0$, $c = 1$ and $p = 3$. If (ii) happens, we have $c = 1$ and $a = d$. By $x_{11} = 0$, we have $a^2 - a + 1 = 0$. Thus the solution of $(x_{ij}) = (y_{ij}) = 0$ are: $a = c = 0$, $b = d = 1$ and $p = 2$; or $a = d = \frac{1 \pm \sqrt{-3}}{2}$, $b = 0$, $c = 1$ and $p = 3$ or $p \equiv 1 (\mathrm{mod}\ \ 6)$. Or equivalently,

$$M_1 = (0, 1, 0, 1)^t,\ \ p = 2;$$

$$M_{21} = (\frac{1 + \sqrt{-3}}{2}, 0, 1, \frac{1 + \sqrt{-3}}{2})^t,\ \ p = 3 \ \mathit{or}\ p \ \equiv \ 1 (\mathrm{mod}\ \ 6);$$

$$M_{22} = (\frac{1 - \sqrt{-3}}{2}, 0, 1, \frac{1 - \sqrt{-3}}{2})^t,\ \ p = 3 \ \mathit{or}\ p \ \equiv \ 1 (\mathrm{mod}\ \ 6).$$

By $X(2, 1)$, $X(p, 1)$ and $X'(p, 1)$, we denote the covering graphs determined by M_1, M_{21} and M_{22}, respectively. In particular, $X(p, 1)$ and $X'(p, 1)$ are the same one if $p = 3$.

Inserting $\gamma = (18)(27)(36)(45)$ to (6), we have

$$\begin{aligned} \mathbf{0} &= D_1^\gamma + D_2^\gamma M = (c_{26})^\gamma + (c_{46}c_{47}c_{37}c_{35})^\gamma M \\ &= (c_{73}) + (c_{53}c_{52}c_{62}c_{64})M \\ &= \begin{pmatrix} b - ab + ac - d \\ b - b^2 + bc \\ b - bc + c^2 - 1 \\ -a + b - bd + cd \end{pmatrix}. \end{aligned}$$

Clearly, the matrices M_1, M_{21} and M_{22} satisfy this equation and γ can be lifted, and so the group N can be lifted.

Inserting $\delta = (23)(67)$ to (6), we have

$$\begin{aligned}
\mathbf{0} &= D_1^\delta + D_2^\delta M = (c_{26})^\delta + (c_{46}c_{47}c_{37}c_{35})^\delta M \\
&= (c_{37}) + (c_{47}c_{46}c_{26}c_{25})M \\
&= \begin{pmatrix} b - ac + ad - d \\ a - bc + bd - d \\ -c^2 + cd - d + 1 \\ -cd + d^2 - d \end{pmatrix}.
\end{aligned}$$

Clearly, the matrix M_1 satisfy this equation and δ can be lifted. But the matrices M_{21} and M_{22} satisfy this equation and δ can be lifted when $p = 3$. So, for graphs $X(2,1)$ and $X(3,1)$, $\mathrm{Aut}(\Gamma)$ can be lifted.

Finally, we show that for $p \equiv 1 (\mathrm{mod}\ \ 6)$, the graphs $X(p,1)$ and $X'(p,1)$ are isomorphic as graphs. Let $V := V(X(p,1)) = V(X'(p,1)) = \{(i;x) \mid 1 \le \mathrm{i} \le 8,\ x \in GF(p)\}$. Let $R = (\frac{-1+\sqrt{-3}}{2})$, and let $\zeta = (18)(25)(36)(47) \in \mathrm{Aut}(\Gamma)$. Define a permutation Υ on V by $(i;x)^\Upsilon = (i^\zeta; (x)R)$. A direct checking shows that Υ is isomorphism from $X(p,1)$ to $X'(p,1)$.

Cases (2) and (3): By a computation similar to case (1), we can get the graph $X(p,1)$ in case (2) and the graph $X(2,1)$ in the case (3).

3.2 The case of $n = 2, 3,$ and 4

In this subsection, the case $n = 2, 3,$ and 4 will be described briefly.

Case $n = 2$. In the case, $K = Z_p^2 = V(2,p)$. As before, without loss we may assume one of the following happens:

(1) $E_1 = \{46, 37\}$ and $E_2 = \{26, 47, 35\}$;

(2) $E_1 = \{26, 46\}$ and $E_2 = \{47, 37, 35\}$;

(3) $E_1 = \{26, 47\}$ and $E_2 = \{46, 37, 35\}$;

(4) $E_1 = \{26, 37\}$ and $E_2 = \{46, 47, 35\}$;

(5) $E_1 = \{26, 35\}$ and $E_2 = \{46, 47, 37\}$;

(6) $E_1 = \{46, 47\}$ and $E_2 = \{26, 37, 35\}$.

For the case of (1), let

$$M = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}.$$

be a voltage generating matrix. Then a computation similar to case $n = 1$ gives that $\mathrm{Aut}(\Gamma)$ can be lifted if and only if $a_1 = a_2 = b_2 = b_3$ and $b_1 = a_3 = 1$. Accordingly, we can get the graph $X(p, 2)$.

In the Cases of (2), (3), (4), (5) and (6), the group M can not be lifted.

Case $n = 3$. In this case, $K = Z_p^3 = V(3, p)$. Without any loss of generality, we may assume one of the following happens:

(1) $E_1 = \{26, 47, 35\}$ and $E_2 = \{46, 37\}$;

(2) $E_1 = \{26, 46, 47\}$ and $E_2 = \{37, 35\}$;

(3) $E_1 = \{26, 46, 37\}$ and $E_2 = \{47, 35\}$;

(4) $E_1 = \{26, 46, 35\}$ and $E_2 = \{47, 37\}$;

(5) $E_1 = \{46, 47, 37\}$ and $E_2 = \{26, 35\}$;

(6) $E_1 = \{26, 47, 37\}$ and $E_2 = \{46, 35\}$.

For the case of (1), one can show that $\mathrm{Aut}(\Gamma)$ can be lifted if and only if the voltage generation matrix

$$M = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Thus, we get the graph $X(p, 3)$.

In the case (2), (3), (4), (5) and (6), the group M can not be lifted.

Case $n = 4$. In this case, $K = Z_p^4 = V(4, p)$. Without any loss of generality, we may divide our discussion into mutually exclusive three cases as followings:

Case (1): $E_1 = \{26, 46, 47, 37\}$ and $E_2 = \{35\}$.

Let $D = (D_0, D_1, D_2)$ be the discriminant matrix with $D_0 = (-M, I_{1\times 1})P$, $D_1 = (-M)$ and $D_2 = (I_{1\times 1})$ with a voltage generating matrix $M = (a, b, c, d)$.

The group $\mathrm{Aut}(\Gamma)$ can be lifted if and only if

$$M_1 = (\frac{1-\sqrt{-3}}{2}, -1, \frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}),\ p = 3 \ \mathit{or}\ p \equiv 1 (\mathrm{mod}\ \ 6);$$

$$M_2 = (\frac{1+\sqrt{-3}}{2}, -1, \frac{1-\sqrt{-3}}{2}, \frac{1+\sqrt{-3}}{2}),\ p = 3 \ \mathit{or}\ p \equiv 1 (\mathrm{mod}\ \ 6).$$

By $X(p, 4)$ and $X'(p, 4)$, we denote the covering graphs determined by M_1 and M_2, respectively. In particular, $X(p, 4)$ and $X'(p, 4)$ are the same one if $p = 3$.

For $p \equiv 1(\mathrm{mod}\ \ 6)$, the graphs $X(p, 4)$ and $X'(p, 4)$ are isomorphic as graphs. Let $V :=$

$V(X(p,4)) = V(X'(p,4)) = \{(i;x,y,z,w) \mid 1 \leq \text{i} \leq 8,\ x,y,z,w \in GF(p)\}$. Let

$$R = \begin{pmatrix} c_2-1 & 1 & -c_2 & c_2-1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

And let and let $\zeta = (18)(25)(36)(47) \in \mathrm{Aut}(\Gamma)$. Define a permutation Υ on V by $(i;x,y,z,w)^{\Upsilon} = (i^{\zeta};(x,y,z,w)R)$. A direct checking shows that Υ is isomorphism from $X(p,4)$ to $X'(p,4)$.

Case (2): $E_1 = \{26, 46, 47, 35\}$ and $E_2 = \{37\}$, and $\{\phi_{26}, \phi_{46},\ \phi_{47}\,\phi_{37}\}$, $\{\phi_{46}, \phi_{47}, \phi_{37}\,\phi_{35}\}$ are linear dependant, hence letting

$$M = (0, b, c, 0).$$

Now, one can show that M cannot be lifted.

By a similar computation, in case (3) , $E_1 = \{26, 46, 37, 35\}$ and $E_2 = \{47\}$, and the group M can not be lifted.

Combining subsection 3.1 and 3.2, we finish the proof of Theorem 1.1. $\square$

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Super Fibonacci Graceful Labeling of Some Special Class of Graphs

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Abstract: A Smarandache-Fibonacci Triple is a sequence $S(n)$, $n \geq 0$ such that $S(n) = S(n-1) + S(n-2)$, where $S(n)$ is the Smarandache function for integers $n \geq 0$. Certainly, it is a generalization of Fibonacci sequence. A Fibonacci graceful labeling and a super Fibonacci graceful labeling on graphs were introduced by Kathiresan and Amutha in 2006. Generally, let G be a (p,q)-graph and $S(n)|n \geq 0$ a Smarandache-Fibonacci Triple. An bijection $f : V(G) \to \{S(0), S(1), S(2), \ldots, S(q)\}$ is said to be a super Smarandache-Fibonacci graceful graph if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{S(1), S(2), \ldots, S(q)\}$. Particularly, if $S(n)$, $n \geq 0$ is just the Fibonacci sequence F_i, $i \geq 0$, such a graph is called a super Fibonacci graceful graph. In this paper, we show that some special class of graphs namely F_n^t, C_n^t and $S_{m,n}^t$ are super fibonacci graceful graphs.

Key Words: Smarandache-Fibonacci triple, graceful labeling, Fibonacci graceful labeling, super Smarandache-Fibonacci graceful graph, super Fibonacci graceful graph.

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§1. Introduction

By a graph, we mean a finite undirected graph without loops or multiple edges. A path of length n is denoted by P_{n+1}. A cycle of length n is denoted by C_n. G^+ is a graph obtained from the graph G by attaching pendant vertex to each vertex of G. Graph labelings, where the vertices are assigned certain values subject to some conditions, have often motivated by practical problems. In the last five decades enormous work has been done on this subject [1].

The concept of graceful labeling was first introduced by Rosa [6] in 1967. A function f is a graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, q\}$ such that when each edge uv is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct.

The notion of Fibonacci graceful labeling and Super Fibonacci graceful labeling were introduced by Kathiresan and Amutha [5]. We call a function f, a fibonacci graceful labeling of a graph G with q edges if f is an injection from the vertices of G to the set $\{0, 1, 2, \ldots, F_q\}$, where

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F_q is the q^{th} fibonacci number of the fibonacci series $F_1 = 1$,$F_2 = 2$,$F_3 = 3$, $F_4 = 5$,$\ldots$, such that each edge uv is assigned the labels $|f(u) - f(v)|$, the resulting edge labels are $F_1, F_2, \ldots, \ldots F_q$. An injective function $f : V(G) \to \{F_0, F_1, \ldots, F_q\}$, where F_q is the q^{th} fibonacci number, is said to be a super fibonacci graceful labeling if the induced edge labeling $|f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$. In the labeling problems the induced labelings must be distinct. So to introduce fibonacci graceful labelings we assume $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots,$ as the sequence of fibonacci numbers instead of $0, 1, 2, \cdots$ [3].

Generally, a Smarandache-Fibonacci Triple is a sequence $S(n)$, $n \geq 0$ such that $S(n) = S(n-1) + S(n-2)$, where $S(n)$ is the Smarandache function for integers $n \geq 0$ ([2]). A (p, q)-graph G is a super Smarandache-Fibonacci graceful graph if there is an bijection $f : V(G) \to \{S(0), S(1), S(2), \ldots, S(q)\}$ such that the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{S(1), S(2), \ldots, S(q)\}$. So a super Fibonacci graceful graph is a special type of Smarandache-Fibonacci graceful graph by definition.

We have constructed some new types of graphs namely $F_n \oplus K^+_{1,m}$, $C_n \oplus P_m$, $K_{1,n} \oslash K_{1,2}$, $F_n \oplus P_m$ and $C_n \oplus K_{1,m}$ and we proved that these graphs are super fibonacci graceful labeling in [7]. In this paper, we prove that F^t_n, C^t_n and $S^t_{m,n}$ are super fibonacci graceful graphs.

§2. Main Results

In this section, we show that some special class of graphs namely F^t_n, C^t_n and $S^t_{m,n}$ are super fibonacci graceful graphs.

Definition 2.1 *Let G be a (p, q) graph. An injective function $f : V(G) \to \{F_0, F_1, F_2, \cdots, F_q\}$, where F_q is the q^{th} fibonacci number, is said to be a super fibonacci graceful graphs if the induced edge labeling $f^*(uv) = |f(u) - f(v)|$ is a bijection onto the set $\{F_1, F_2, \ldots, F_q\}$.*

Definition 2.2 *The one point union of t copies of fan F_n is denoted by F^t_n.*

The following theorem shows that the graph F^t_n is a super fibonacci graceful graph.

Theorem 2.3 *F^t_n is a super fibonacci graceful graph for all $n \geq 2$.*

Proof Let u_0 be the center vertex of F^t_n and u^j_i, where $i = 1, 2, \ldots, t$, $j = 1, 2, \ldots, n$ be the other vertices of F^t_n. Also, $|V(G)| = nt + 1$ and $|E(G)| = 2nt - t$. Define $f : V(F^t_n) \to \{F_0, F_1, \ldots, F_q\}$ by $f(u_0) = F_0$, $f(u^j_1) = F_{2j-1}$, $1 \leq j \leq n$. For $i = 2, 3, \ldots, t$, $f(u^j_i) = F_{2n(i-1)+2(j-1)-(i-2)}$, $1 \leq j \leq n$. We claim that all these edge labels are distinct. Let $E_1 = \{f^*(u_0u^j_1) : 1 \leq j \leq n\}$. Then

$$\begin{aligned} E_1 &= \{|f(u_0) - f(u^j_1)| : 1 \leq j \leq n\} \\ &= \{|f(u_0) - f(u^1_1)|, |f(u_0) - f(u^2_1)|, \ldots, |f(u_0) - f(u^{n-1}_1)|, |f(u_0) - f(u^n_1)|\} \\ &= \{|F_0 - F_1|, |F_0 - F_3|, \ldots, |F_0 - F_{2n-3}|, |F_0 - F_{2n-1}|\} \\ &= \{F_1, F_3, \ldots, F_{2n-3}, F_{2n-1}\}. \end{aligned}$$

Let $E_2 = \{f^*(u_1^j u_1^{j+1}) : 1 \leq j \leq n-1\}$. Then

$$\begin{aligned} E_2 &= \{|f(u_1^j) - f(u_1^{j+1})| : 1 \leq j \leq n-1\} \\ &= \{|f(u_1^1) - f(u_1^2)|, |f(u_1^2) - f(u_1^3)|, \ldots, |f(u_1^{n-2}) - f(u_1^{n-1})|, |f(u_1^{n-1}) - f(u_1^n)|\} \\ &= \{|F_1 - F_3|, |F_3 - F_5|, \ldots, |F_{2n-5} - F_{2n-3}|, |F_{2n-3} - F_{2n-1}|\} \\ &= \{F_2, F_4, \ldots, F_{2n-4}, F_{2n-2}\}. \end{aligned}$$

For $i = 2$, we know that

$$\begin{aligned} E_3 &= \{f^*(u_0 u_2^j) : 1 \leq j \leq n\} = \{|f(u_0) - f(u_2^j)| : 1 \leq j \leq n\} \\ &= \{|f(u_0) - f(u_2^1)|, |f(u_0) - f(u_2^2)|, \ldots, |f(u_0) - f(u_2^{n-1})|, |f(u_0) - f(u_2^n)|\} \\ &= \{|F_0 - F_{2n}|, |F_0 - F_{2n+2}|, \ldots, |F_0 - F_{4n-4}|, |F_0 - F_{4n-2}|\} \\ &= \{F_{2n}, F_{2n+2}, \ldots, F_{4n-4}, F_{4n-2}\}, \end{aligned}$$

$$\begin{aligned} E_4 &= \{f^*(u_2^j u_2^{j+1}) : 1 \leq j \leq n-1\} \\ &= \{|f(u_2^j) - f(u_2^{j+1})| : 1 \leq j \leq n-1\} \\ &= \{|f(u_2^1) - f(u_2^2)|, |f(u_2^2) - f(u_2^3)|, \ldots, |f(u_2^{n-2}) - f(u_2^{n-1})|, |f(u_2^{n-1}) - f(u_2^n)|\} \\ &= \{|F_{2n} - F_{2n+2}|, |F_{2n+2} - F_{2n+4}|, \ldots, |F_{4n-6} - F_{4n-4}|, |F_{4n-4} - F_{4n-2}|\} \\ &= \{F_{2n+1}, F_{2n+3}, \ldots, F_{4n-5}, F_{4n-3}\}. \end{aligned}$$

For $i = 3$, let $E_5 = \{f^*(u_0 u_3^j) : 1 \leq j \leq n\}$. Then

$$\begin{aligned} E_5 &= \{|f(u_0) - f(u_3^j)| : 1 \leq j \leq n\} \\ &= \{|f(u_0) - f(u_3^1)|, |f(u_0) - f(u_3^2)|, \ldots, |f(u_0) - f(u_3^{n-1})|, |f(u_0) - f(u_3^n)|\} \\ &= \{|F_0 - F_{4n-1}|, |F_0 - F_{4n+1}|, \ldots, |F_0 - F_{6n-5}|, |F_0 - F_{6n-3}|\} \\ &= \{F_{4n-1}, F_{4n+1}, \ldots, F_{6n-5}, F_{6n-3}\}. \end{aligned}$$

Let $E_6 = \{f^*(u_3^j u_3^{j+1}) : 1 \leq j \leq n-1\}$. Then

$$\begin{aligned} E_6 &= \{|f(u_3^j) - f(u_3^{j+1})| : 1 \leq j \leq n-1\} \\ &= \{|f(u_3^1) - f(u_3^2)|, |f(u_3^2) - f(u_3^3)|, \ldots, |f(u_3^{n-2}) - f(u_3^{n-1})|, |f(u_3^{n-1}) - f(u_3^n)|\} \\ &= \{|F_{4n-1} - F_{4n+1}|, |F_{4n+1} - F_{4n+3}|, \ldots, |F_{6n-7} - F_{6n-5}|, |F_{6n-5} - F_{6n-3}|\} \\ &= \{F_{4n}, F_{4n+2}, \ldots, F_{6n-6}, F_{6n-4}\} \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots ,$$

Now, for $i = t-1$, let $E_{t-1} = \{f^*(u_0 u_{t-1}^j) : 1 \leq j \leq n\}$. Then

$$\begin{aligned} E_{t-1} &= \{|f(u_0) - f(u_{t-1}^j)| : 1 \leq j \leq n\} \\ &= \{|f(u_0) - f(u_{t-1}^1)|, |f(u_0) - f(u_{t-1}^2)|, \ldots, |f(u_0) - f(u_{t-1}^{n-1})|, |f(u_0) - f(u_{t-1}^n)|\} \\ &= \{|F_0 - F_{2nt-4n-t+3}|, |F_0 - F_{2nt-4n-t+5}|, \ldots, |F_0 - F_{2nt-2n-t-1}|, |F_0 - F_{2nt-2n-t+1}|\} \\ &= \{F_{2nt-4n-t+3}, F_{2nt-4n-t+5}, \ldots, F_{2nt-2n-t-1}, F_{2nt-2n-t+1}\}. \end{aligned}$$

Let $E_{t-1} = \{f^*(u_{t-1}^{j}u_{t-1}^{j+1}) : 1 \le j \le n-1\}$. Then

$$\begin{aligned}
E_{t-1} &= \{|f(u_{t-1}^{j}) - f(u_{t-1}^{j+1})| : 1 \le j \le n-1\}\\
&= \{|f(u_{t-1}^{1}) - f(u_{t-1}^{2})|, |f(u_{t-1}^{2}) - f(u_{t-1}^{3})|,\\
&\quad \ldots, |f(u_{t-1}^{n-2}) - f(u_{t-1}^{n-1})|, |f(u_{t-1}^{n-1}) - f(u_{t-1}^{n})|\}\\
&= \{|F_{2nt-4n-t+3} - F_{2nt-4n-t+5}|, |F_{2nt-4n-t+5} - F_{2nt-4n-t+7}|,\\
&\quad \ldots, |F_{2nt-2n-t-3} - F_{2nt-2n-t-1}|, |F_{2nt-2n-t-1} - F_{2nt-2n-t+1}|\}\\
&= \{F_{2nt-4n-t+4}, F_{2nt-4n-t+6}, \ldots, F_{2nt-2n-t-2}, F_{2nt-2n-t}\}.
\end{aligned}$$

F_4^4 :

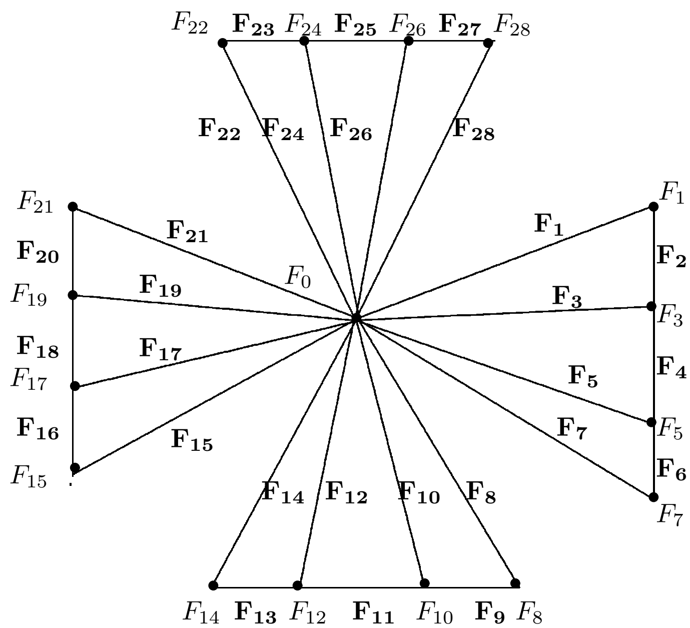

Fig.1

For $i = t$, let $E_t = \{f^*(u_0 u_t^{j}) : 1 \le j \le n\}$. Then

$$\begin{aligned}
E_t &= \{|f(u_0) - f(u_t^{j})| : 1 \le j \le n\}\\
&= \{|f(u_0) - f(u_t^{1})|, |f(u_0) - f(u_t^{2})|, \ldots, |f(u_0) - f(u_t^{n-1})|,\\
&\quad |f(u_0) - f(u_t^{n})|\}\\
&= \{|F_0 - F_{2nt-2n-t+2}|, |F_0 - F_{2nt-2n-t+4}|, \ldots, |F_0 - F_{2nt-t-2}|, |F_0 - F_{2nt-t}|\}\\
&= \{F_{2nt-2n-t+2}, F_{2nt-2n-t+4}, \ldots, F_{2nt-t-2}, F_{2nt-t}\}.
\end{aligned}$$

Let $E_t = \{f^*(u_t^j u_t^{j+1}) : 1 \leq j \leq n-1\}$. Then

$$\begin{aligned} E_t &= \{|f(u_t^j) - f(u_t^{j+1})| : 1 \leq j \leq n-1\} \\ &= \{|f(u_t^1) - f(u_t^2)|, |f(u_t^2) - f(u_t^3)|, \ldots, |f(u_t^{n-2}) - f(u_t^{n-1})|, |f(u_t^{n-1}) - f(u_t^n)|\} \\ &= \{|F_{2nt-2n-t+2} - F_{2nt-2n-t+4}|, |F_{2nt-2n-t+4} - F_{2nt-2n-t+6}|, \\ &\quad \ldots, |F_{2nt-t-4} - F_{2nt-t-2}|, |F_{2nt-t-2} - F_{2nt-t}|\} \\ &= \{F_{2nt-2n-t+3}, F_{2nt-2n-t+5}, \ldots, F_{2nt-t-3}, F_{2nt-t-1}\}. \end{aligned}$$

Therefore, $E = E_1 \cup E_2 \cup, \ldots, \cup E_{t-1} \cup E_t = \{F_1, F_2, \ldots, F_{2nt-t}\}$ Thus, the edge labels are distinct. Therefore, F_n^t admits super fibonacci graceful labeling. □

For example the super fibonacci graceful labeling of F_4^4 is shown in Fig.1.

Definition 2.4 *The one point union of t cycles of length n is denoted by C_n^t.*

Theorem 2.5 *C_n^t is a super fibonacci graceful graph for $n \equiv 0(mod3)$.*

Proof Let u_0 be the one point union of t cycles and $u_1, u_2, \ldots, u_{t(n-1)}$ be the other vertices of C_n^t. Also, $|V(G)| = t(n-1)+1, |E(G)| = nt$. Define $f : V(C_n^t) \to \{F_0, F_1, \ldots, F_q\}$ by $f(u_0) = F_0$. For $i = 1, 2, \ldots, t$, $f(u_{(n-1)(i-1)+(j-1)+1}) = F_{nt-n(i-1)-2(j-1)}$, $1 \leq j \leq 2$. For $s = 1, 2, \cdots, \frac{n-3}{3}$, $i = 1, 2, \ldots, t$, $f(u_{(n-1)(i-1)+j}) = F_{nt-1-n(i-1)-2(j-s-2)+(s-1)}$, $3s \leq j \leq 3s+2$. Next, we claim that the edge labels are distinct.

We find the edge labeling between the vertex u_0 and starting vertex of each copy of $(u_{(n-1)(i-1)+1})$. Let $E_1 = \{f^*(u_0 u_{(n-1)(i-1)+1}) : 1 \leq i \leq t\}$. Then

$$\begin{aligned} E_1 &= \{|f(u_0) - f(u_{(n-1)(i-1)+1})| : 1 \leq i \leq t\} \\ &= \{|f(u_0) - f(u_1)|, |f(u_0) - f(u_n)|, \ldots, |f(u_0) - f(u_{nt-2n-t+3})|, \\ &\quad |f(u_0) - f(u_{nt-n-t+2})|\} \\ &= \{|F_0 - F_{nt}|, |F_0 - F_{nt-n}|, \ldots, |F_0 - F_{2n}|, |F_0 - F_n|\} \\ &= \{F_{nt}, F_{nt-n}, \ldots, F_{2n}, F_n\} \end{aligned}$$

Now we determine the edge labelings between the vertex $u_{(n-1)(i-1)+1}$ and the vertex $u_{(n-1)(i-1)+2}$ of each copy. Let $E_2 = \{f^*(u_{(n-1)(i-1)+1} u_{(n-1)(i-1)+2}) : 1 \leq i \leq t\}$. Then

$$\begin{aligned} E_2 &= \{|f(u_{(n-1)(i-1)+1}) - f(u_{(n-1)(i-1)+2})| : 1 \leq i \leq t\} \\ &= \{|f(u_1) - f(u_2)|, |f(u_n) - f(u_{n+1})|, \ldots, \\ &\quad |f(u_{nt-2n-t+3}) - f(u_{nt-2n-t+4})|, |f(u_{nt-n-t+2}) - f(u_{nt-n-t+3})|\} \\ &= \{|F_{nt} - F_{nt-2}|, |F_{nt-n} - F_{nt-n-2}|, \ldots, |F_{2n} - F_{2n-2}|, |F_n - F_{n-2}|\} \\ &= \{F_{nt-1}, F_{nt-n-1}, \ldots, F_{2n-1}, F_{n-1}\} \end{aligned}$$

We calculate the edge labeling between the vertex $u_{(n-1)(i-1)+2}$ and starting vertex $u_{(n-1)(i-1)+3}$

of the first loop. Let $E_3 = \{f^*(u_{(n-1)(i-1)+2}u_{(n-1)(i-1)+3}) : 1 \leq i \leq t\}$. Then

$$\begin{aligned}
E_3 &= \{|f(u_{(n-1)(i-1)+2}) - f(u_{(n-1)(i-1)+3})| : 1 \leq i \leq t\} \\
&= \{|f(u_2) - f(u_3)|, |f(u_{n+1}) - f(u_{n+2})|, |f(u_{2n}) - f(u_{2n+1})|, \ldots, \\
&\quad |f(u_{nt-2n-t+4}) - f(u_{nt-2n-t+5})|, |f(u_{nt-n-t+3}) - f(u_{nt-n-t+4})|\} \\
&= \{|F_{nt-2} - F_{nt-1}|, |F_{nt-n-2} - F_{nt-n-1}|, |F_{nt-2n-2} - F_{nt-2n-1}|, \\
&\quad \ldots, |F_{2n-2} - F_{2n-1}|, |F_{n-2} - F_{n-1}|\} \\
&= \{F_{nt-3}, F_{nt-n-3}, F_{nt-2n-3}, \ldots, F_{2n-3}, F_{n-3}\}.
\end{aligned}$$

Now, for $s = 1$, let $E_4 = \bigcup_{i=1}^{t} \{f^*(u_{(n-1)(i-1)+j}u_{(n-1)(i-1)+j+1}) : 3 \leq j \leq 4\}$. Then

$$\begin{aligned}
E_4 &= \cup_{i=1}^{t} \{|f(u_{(n-1)(i-1)+j}) - f(u_{(n-1)(i-1)+j+1})| : 3 \leq j \leq 4\} \\
&= \{|f(u_3) - f(u_4)|, |f(u_4) - f(u_5)|\} \\
&\quad \cup \{|f(u_{n+2}) - f(u_{n+3})|, |f(u_{n+3}) - f(u_{n+4})|\} \cup, \ldots, \\
&\quad \cup \{|f(u_{nt-2n-t+5}) - f(u_{nt-2n-t+6})|, |f(u_{nt-2n-t+6}) - f(u_{nt-2n-t+7})|\} \\
&\quad \cup \{|f(u_{nt-n-t+4}) - f(u_{nt-n-t+5})|, |f(u_{nt-n-t+5}) - f(u_{nt-n-t+6})|\} \\
&= \{|F_{nt-1} - F_{nt-3}|, |F_{nt-3} - F_{nt-5}|\} \\
&\quad \cup \{|F_{nt-n-1} - F_{nt-n-3}|, |F_{nt-n-3} - F_{nt-n-5}|\} \\
&\quad \cup, \ldots, \cup \{|F_{2n-1} - F_{2n-3}|, |F_{2n-3} - F_{2n-5}|\} \\
&\quad \cup \{|F_{n-1} - F_{n-3}|, |F_{n-3} - F_{n-5}|\} \\
&= \{F_{nt-2}, F_{nt-4}\} \cup \{F_{nt-n-2}, F_{nt-n-4}\} \cup, \ldots, \\
&\quad \cup \{F_{2n-2}, F_{2n-4}\} \cup \{F_{n-2}, F_{n-4}\}
\end{aligned}$$

For the edge labeling between the end vertex $(u_{(n-1)(i-1)+5})$ of the first loop and starting vertex $(u_{(n-1)(i-1)+6})$ of the second loop, calculation shows that

$$\begin{aligned}
E_4^1 &= \{|f(u_{(n-1)(i-1)+5}) - f(u_{(n-1)(i-1)+6})| : 1 \leq i \leq t\} \\
&= \{|f(u_5) - f(u_6)|, |f(u_{n+4}) - f(u_{n+5})|, \ldots, \\
&\quad |f(u_{nt-2n-t+7}) - f(u_{nt-2n-t+8})|, |f(u_{nt-n-t+6}) - f(u_{nt-n-t+7})|\} \\
&= \{|F_{nt-5} - F_{nt-4}|, |F_{nt-n-5} - F_{nt-n-4}|, \ldots, |F_{2n-5} - F_{2n-4}|, |F_{n-5} - F_{n-4}|\} \\
&= \{F_{nt-6}, F_{nt-n-6}, \ldots, F_{2n-6}, F_{n-6}\}
\end{aligned}$$

For $s = 2$, let $E_5 = \cup_{i=1}^{t} \{f^*(u_{(n-1)(i-1)+j}u_{(n-1)(i-1)+j+1}) : 6 \leq j \leq 7\}$. Then

$$\begin{aligned}
E_5 &= \cup_{i=1}^{t}\{|f(u_{(n-1)(i-1)+j}) - f(u_{(n-1)(i-1)+j+1})| : 6 \le j \le 7\} \\
&= \{|f(u_6) - f(u_7)|, |f(u_7) - f(u_8)|\} \\
&\quad \cup\{|f(u_{n+5}) - f(u_{n+6})|, |f(u_{n+6}) - f(u_{n+7})|\}\cup, \ldots, \\
&\quad \cup|f(u_{nt-2n-t+8}) - f(u_{nt-2n-t+9})|, |f(u_{nt-2n-t+9}) - f(u_{nt-2n-t+10})|\} \\
&\quad \cup\{|f(u_{nt-n-t+7}) - f(u_{nt-n-t+8})|, |f(u_{nt-n-t+8}) - f(u_{nt-n-t+9})|\} \\
&= \{|F_{nt-4} - F_{nt-6}|, |F_{nt-6} - F_{nt-8}|\} \\
&\quad \cup\{|F_{nt-n-4} - F_{nt-n-6}|, |F_{nt-n-6} - F_{nt-n-8}|\} \\
&\quad \cup, \ldots, \cup\{|F_{2n-4} - F_{2n-6}|, |F_{2n-6} - F_{2n-8}|\} \\
&\quad \cup\{|F_{n-4} - F_{n-6}|, |F_{n-6} - F_{n-8}|\} \\
&= \{F_{nt-5}, F_{nt-7}\} \cup \{F_{nt-n-5}, F_{nt-n-7}\}\cup, \ldots, \\
&\quad \cup\{F_{2n-5}, F_{2n-7}\} \cup \{F_{n-5}, F_{n-7}\}
\end{aligned}$$

Similarly, for finding the edge labeling between the end vertex $(u_{(n-1)(i-1)+8})$ of the second loop and starting vertex $(u_{(n-1)(i-1)+9})$ of the third loop, calculation shows that

$$\begin{aligned}
E_5^1 &= \{|f(u_{(n-1)(i-1)+8}) - f(u_{(n-1)(i-1)+9})| : 1 \le i \le t\} \\
&= \{|f(u_8) - f(u_9)|, |f(u_{n+7}) - f(u_{n+8})|, \ldots, \\
&\quad |f(u_{nt-2n-t+10}) - f(u_{nt-2n-t+11})|, |f(u_{nt-n-t+9}) - f(u_{nt-n-t+10})|\} \\
&= \{|F_{nt-8} - F_{nt-7}|, |F_{nt-n-8} - F_{nt-n-7}|, \ldots, |F_{2n-8} - F_{2n-7}|, \\
&\quad |F_{n-8} - F_{n-7}|\} \\
&= \{F_{nt-9}, F_{nt-n-9}, \ldots, F_{2n-9}, F_{n-9}\},
\end{aligned}$$

$$\cdots\cdots\cdots\cdots,$$

For $s = \dfrac{n-3}{3} - 1$, let $E_{\frac{n-3}{3}-1} = \cup_{i=1}^{t}\{f^*(u_{(n-1)(i-1)+j}u_{(n-1)(i-1)+j+1}) : n-6 \le j \le n-5\}$. Then

$$\begin{aligned}
E_{\frac{n-3}{3}-1} &= \cup_{i=1}^{t}\{|f(u_{(n-1)(i-1)+j}) - f(u_{(n-1)(i-1)+j+1})| : n-6 \le j \le n-5\} \\
&= \{|f(u_{n-6}) - f(u_{n-5})|, |f(u_{n-5}) - f(u_{n-4})|\} \\
&\quad \cup\{|f(u_{2n-7}) - f(u_{2n-6})|, |f(u_{2n-6}) - f(u_{2n-5})|\}\cup, \ldots, \\
&\quad \cup\{|f(u_{nt-n-t-4}) - f(u_{nt-n-t-3})|, |f(u_{nt-n-t-3}) - f(u_{nt-n-t-2})|\} \\
&\quad \cup\{|f(u_{nt-t-5}) - f(u_{nt-t-4})|, |f(u_{nt-t-4}) - f(u_{nt-t-3})|\} \\
&= \{|F_{nt-n+8} - F_{nt-n+6}|, |F_{nt-n+6} - F_{nt-n+4}|\} \cup \{|F_{nt-2n+8} - F_{nt-2n+6}|, \\
&\quad |F_{nt-2n+6} - F_{nt-2n+4}|\}\cup, \ldots, \cup\{|F_{n+8} - F_{n+6}|, |F_{n+6} - F_{n+4}|\} \\
&\quad \cup\{|F_8 - F_6|, |F_6 - F_4|\} \\
&= \{F_{nt-n+7}, F_{nt-n+5}\} \cup \{F_{nt-2n+7}, F_{nt-2n+5}\}\cup, \ldots, \cup\{F_{n+7}, F_{n+5}\} \\
&\quad \cup\{F_7, F_5\}
\end{aligned}$$

We calculate the edge labeling between the end vertex $(u_{(n-1)(i-1)+n-4})$ of the $(\frac{n-3}{3}-1)^{th}$ loop and starting vertex $(u_{(n-1)(i-1)+n-3})$ of the $(\frac{n-3}{3})^{rd}$ loop as follows.

$$\begin{aligned}
E^1_{\frac{n-3}{3}-1} &= \{|f(u_{(n-1)(i-1)+n-4}) - f(u_{(n-1)(i-1)+n-3})| : 1 \le i \le t\} \\
&= \{|f(u_{n-4}) - f(u_{n-3})|, |f(u_{2n-5}) - f(u_{2n-4})|, \ldots, \\
&\quad |f(u_{nt-n-t+2}) - f(u_{nt-n-t-1})|, |f(u_{nt-t-3}) - f(u_{nt-t-2})|\} \\
&= \{|F_{nt-n+6} - F_{nt-n+5}|, |F_{nt-2n+4} - F_{nt-2n+5}|, \ldots, \\
&\quad |F_{n+4} - F_{n+5}|, |F_4 - F_5|\} \\
&= \{F_{nt-n+4}, F_{nt-2n+3}, \ldots, F_{n+3}, F_3\}
\end{aligned}$$

For $s = \dfrac{n-3}{3}$, let $E_{\frac{n-3}{3}} = \bigcup_{i=1}^{t} \{f^*(u_{(n-1)(i-1)+j}u_{(n-1)(i-1)+j+1}) : n-3 \le j \le n-2\}$. Then

$$\begin{aligned}
E_{\frac{n-3}{3}} &= \cup_{i=1}^{t} \{|f(u_{(n-1)(i-1)+j}) - f(u_{(n-1)(i-1)+j+1})| : n-3 \le j \le n-2\} \\
&= \{|f(u_{n-3}) - f(u_{n-2})|, |f(u_{n-2}) - f(u_{n-1})|\} \\
&\quad \cup \{|f(u_{2n-4}) - f(u_{2n-3})|, |f(u_{2n-3}) - f(u_{2n-2})|\} \cup, \ldots, \\
&\quad \cup |f(u_{nt-n-t-1}) - f(u_{nt-n-t})|, |f(u_{nt-n-t}) - f(u_{nt-n-t+1})|\} \\
&\quad \cup \{|f(u_{nt-t-2}) - f(u_{nt-t-1})|, |f(u_{nt-t-1}) - f(u_{nt-t})|\} \\
&= \{|F_{nt-n+5} - F_{nt-n+3}|, |F_{nt-n+3} - F_{nt-n+1}|\} \\
&\quad \cup \{|F_{nt-2n+5} - F_{nt-2n+3}|, |F_{nt-2n+3} - F_{nt-2n+1}|\} \cup, \ldots, \\
&\quad \cup \{|F_{n+5} - F_{n+3}|, |F_{n+3} - F_{n+1}|\} \cup \{|F_5 - F_3|, |F_3 - F_1|\} \\
&= \{F_{nt-n+4}, F_{nt-n+2}\} \cup \{F_{nt-2n+4}, F_{nt-2n+2}\} \cup, \ldots, \\
&\quad \cup \{F_{n+4}, F_{n+2}\} \cup \{F_4, F_2\}
\end{aligned}$$

Calculation shows the edge labeling between the end vertex $(u_{(n-1)(i-1)+n-1})$ of the $(\frac{n-3}{3})^{rd}$ loop and the vertex u_0 are

$$\begin{aligned}
E^* &= \{|f(u_{(n-1)(i-1)+n-1}) - f(u_0)| : 1 \le i \le t\} \\
&= \{|f(u_{n-1}) - f(u_0)|, |f(u_{2n-2}) - f(u_0)|, \ldots, \\
&\quad |f(u_{nt-n-t+1}) - f(u_0)|, |f(u_{nt-t}) - f(u_0)|\} \\
&= \{|F_{nt-n+1} - F_0|, |F_{nt-2n+1} - F_0|, \ldots, |F_{n+1} - F_0|, |F_1 - F_0|\} \\
&= \{F_{nt-n+1}, F_{nt-2n+1}, \ldots, F_{n+1}, F_1\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E &= (E_1 \cup E_2 \cup, \ldots, \cup E_{\frac{n-3}{3}}) \cup (E^1_4 \cup E^1_5 \cup, \ldots, \cup E^1_{\frac{n-3}{3}-1}) \cup E^* \\
&= \{F_1, F_2, \ldots, F_{nt}\}
\end{aligned}$$

Thus, the edge labels are distinct. Therefore, C^t_n admits a super fibonacci graceful labeling. $\square$

C_9^2 :

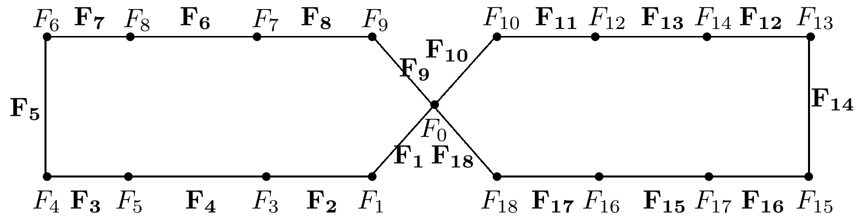

Fig.2

For example the super fibonacci graceful labeling of C_9^2 is shown in Fig.2.

Definition 2.6([4]) *Let $S_{m,n}$ stand for a star with n spokes in which each spoke is a path of length m.*

Definition 2.7 *The one point union of t copies of $S_{m,n}$ is denoted by $S_{m,n}^t$.*

Next theorem shows that the graph $S_{m,n}^t$ is a super Fibonacci graceful graph.

Theorem 2.8 *$S_{m,n}^t$ is a super fibonacci graceful graph for all m,n, when $n \equiv 1(mod3)$.*

Proof Let v_0 be the center of the star and v_j^i, $i = 1,2,\ldots,mt$, $j = 1,2,\ldots,n$ be the other vertices of $S_{m,n}^t$. Also, $|V(G)| = mnt+1$ and $|E(G)| = mnt$. Define $f : V(S_{m,n}^t) \to \{F_0, F_1, \ldots, F_q\}$ by $f(v_0) = F_0$. For $i = 1,2,\ldots,mt$, $f(v_j^i) = F_{mnt-n(i-1)-2(j-1)}$, $1 \le j \le 2$. For $i = 1,2,\ldots,mt$, $F(v_j^i) = F_{mnt-(2j-n-1)-n(i-1)}$, $n-1 \le j \le n$. For $s = 1,2,\ldots,\frac{n-4}{3}$, $i = 1,2,\ldots,mt$
$f(v_j^i) = F_{mnt-1-n(i-1)-2(j-s-2)+(s-1)}$, $3s \le j \le 3s+2$. We claim that all these edge labels are distinct. Let $E_1 = \{f^*(v_0v_1^i) : 1 \le i \le mt\}$. Calculation shows that

$$\begin{aligned}
E_1 &= \{|f(v_0) - f(v_1^i)| : 1 \le i \le mt\} \\
&= \{|f(v_0) - f(v_1^1)|, |f(v_0) - f(v_1^2)|, \ldots, |f(v_0) - f(v_1^{mt-1})|, \\
&\quad |f(v_0) - f(v_1^{mt})|\} \\
&= \{|F_0 - F_{mnt}|, |F_0 - F_{mnt-n}|, \ldots, |F_0 - F_{2n}|, |F_0 - F_n|\} \\
&= \{F_{mnt}, F_{mnt-n}, \ldots, F_{2n}, F_n\}.
\end{aligned}$$

Let $E_2 = \{f^*(v_1^iv_2^i) : 1 \le i \le mt\}$. Then

$$\begin{aligned}
E_2 &= \{|f(v_1^i) - f(v_2^i)| : 1 \le i \le mt\} \\
&= \{|f(v_1^1) - f(v_2^1)|, |f(v_1^2) - f(v_2^2)|, \ldots, |f(v_1^{mt-1}) - f(v_2^{mt-1})|, \\
&\quad |f(v_1^{mt}) - f(v_2^{mt})|\} \\
&= \{|F_{mnt} - F_{mnt-2}|, |F_{mnt-n} - F_{mnt-n-2}|, \ldots, |F_{2n} - F_{2n-2}|, |F_n - F_{n-2}|\} \\
&= \{F_{mnt-1}, F_{mnt-n-1}, \ldots, F_{2n-1}, F_{n-1}\}
\end{aligned}$$

For the edge labeling between the vertex v_2^i and starting vertex v_3^i of the first loop, let $E_3 = \{f^*(v_2^i v_3^i) : 1 \leq i \leq mt\}$. Calculation shows that

$$\begin{aligned}
E_3 &= \{|f(v_2^i) - f(v_3^i)| : 1 \leq i \leq mt\} \\
&= \{|f(v_2^1) - f(v_3^1)|, |f(v_2^2) - f(v_3^2)|, \ldots, |f(v_2^{mt-1}) - f(v_3^{mt-1})|, \\
&\quad |f(v_2^{mt}) - f(v_3^{mt})|\} \\
&= \{|F_{mnt-2} - F_{mnt-1}|, |F_{mnt-n-2} - F_{mnt-n-1}|, \ldots, |F_{2n-2} - F_{2n-1}|, \\
&\quad |F_{n-2} - F_{n-1}|\} \\
&= \{F_{mnt-3}, F_{mnt-n-3}, \ldots, F_{2n-3}, F_{n-3}\}.
\end{aligned}$$

For $s = 1$, let $E_4 = \cup_{i=1}^{mt}\{f^*(v_j^i v_{j+1}^i) : 3 \leq j \leq 4\}$. Then

$$\begin{aligned}
E_4 &= \cup_{i=1}^{mt}\{|f(v_j^i) - f(v_{j+1}^i)| : 3 \leq j \leq 4\} \\
&= \{|f(v_3^1) - f(v_4^1)|, |f(v_4^1) - f(v_5^1)|\} \cup \{|f(v_3^2) - f(v_4^2)|, |f(v_4^2) - f(v_5^2)|\} \\
&\quad \cup, \ldots, \cup\{|f(v_3^{mt-1}) - f(v_4^{mt-1})|, |f(v_4^{mt-1}) - f(v_5^{mt-1})|\} \\
&\quad \cup\{|f(v_3^{mt}) - f(v_4^{mt})|, |f(v_4^{mt}) - f(v_5^{mt})|\} \\
&= \{|F_{mnt-1} - F_{mnt-3}|, |F_{mnt-3} - F_{mnt-5}|\} \\
&\quad \cup\{|F_{mnt-n-1} - F_{mnt-n-3}|, |F_{mnt-n-3} - F_{mnt-n-5}|\} \\
&\quad \cup, \ldots, \cup\{|F_{2n-1} - F_{2n-3}|, |F_{2n-3} - F_{2n-5}|\} \\
&\quad \cup\{|F_{n-1} - F_{n-3}|, |F_{n-3} - F_{n-5}|\} \\
&= \{F_{mnt-2}, F_{mnt-4}\} \cup \{F_{mnt-n-2}, F_{mnt-n-4}\}\cup, \ldots, \cup\{F_{2n-2}, F_{2n-4}\} \\
&\quad \cup\{F_{n-2}, F_{n-4}\}.
\end{aligned}$$

We find the edge labeling between the vertex v_5^i of the first loop and starting vertex v_6^i of the second loop. Let $E_4^1 = \{f^*(v_5^i v_6^i) : 1 \leq i \leq mt\}$. Then

$$\begin{aligned}
E_4^1 &= \{|f(v_5^i) - f(v_6^i)| : 1 \leq i \leq mt\} \\
&= \{|f(v_5^1) - f(v_6^1)|, |f(v_5^2) - f(v_6^2)|, \ldots, |f(v_5^{mt-1}) - f(v_6^{mt-1})|, \\
&\quad |f(v_5^{mt}) - f(v_6^{mt})|\} \\
&= \{|F_{mnt-5} - F_{mnt-4}|, |F_{mnt-n-5} - F_{mnt-n-4}|, \ldots, |F_{2n-5} - F_{2n-4}|, \\
&\quad |F_{n-5} - F_{n-4}|\} \\
&= \{F_{mnt-6}, F_{mnt-n-6}, \ldots, F_{2n-6}, F_{n-6}\}.
\end{aligned}$$

For $s = 2$, let $E_5 = \cup_{i=1}^{mt}\{f^*(v_j^i v_{j+1}^i) : 6 \leq j \leq 7\}$. Then

$$\begin{aligned}
E_5 &= \cup_{i=1}^{mt}\{|f(v_j^i)-f(v_{j+1}^i)| : 6\le j\le 7\}\\
&= \{|f(v_6^1)-f(v_7^1)|, |f(v_7^1)-f(v_8^1)|\}\cup\{|f(v_6^2)-f(v_7^2)|, |f(v_7^2)-f(v_8^2)|\}\\
&\quad \cup,\ldots,\ldots,\{|f(v_6^{mt-1})-f(v_7^{mt-1})|, |f(v_7^{mt-1})-f(v_8^{mt-1})|\}\\
&\quad \cup\{|f(v_6^{mt})-f(v_7^{mt})|, |f(v_7^{mt})-f(v_8^{mt})|\}\\
&= \{|F_{mnt-4}-F_{mnt-6}|, |F_{mnt-6}-F_{mnt-8}|\}\cup\{|F_{mnt-n-4}-F_{mnt-n-6}|,\\
&\quad |F_{mnt-n-6}-F_{mnt-n-8}|\}\cup,\ldots,\cup\{|F_{2n-4}-F_{2n-6}|, |F_{2n-6}-F_{2n-8}|\}\\
&\quad \cup\{|F_{n-4}-F_{n-6}|, |F_{n-6}-F_{n-8}|\}\\
&= \{F_{mnt-5}, F_{mnt-7}\}\cup\{F_{mnt-n-5}, F_{mnt-n-7}\}\cup,\ldots,\cup\{F_{2n-5}, F_{2n-7}\}\\
&\quad \cup\{F_{n-5}, F_{n-7}\}.
\end{aligned}$$

Let $E_5^1=\{f^*(v_8^iv_9^i) : 1\le i\le mt\}$. Calculation shows that the edge labeling between the vertex v_8^i of the second loop and starting vertex v_9^i of the third loop are

$$\begin{aligned}
E_5^1 &= \{|f(v_8^i)-f(v_9^i)| : 1\le i\le mt\}\\
&= \{|f(v_8^1)-f(v_9^1)|, |f(v_8^2)-f(v_9^2)|,\ldots,|f(v_8^{mt-1})-f(v_9^{mt-1})|,\\
&\quad |f(v_8^{mt})-f(v_9^{mt})|\}\\
&= \{|F_{mnt-8}-F_{mnt-7}|, |F_{mnt-n-8}-F_{mnt-n-7}|,\ldots,\\
&\quad |F_{2n-8}-F_{2n-7}|, |F_{n-8}-F_{n-7}|\}\\
&= \{F_{mnt-9}, F_{mnt-n-9},\ldots,F_{2n-9}, F_{n-9}\}.
\end{aligned}$$

For $s=\dfrac{n-4}{3}-1$, let $E_{\frac{n-4}{3}-1}=\cup_{i=1}^{mt}\{f^*(v_j^iv_{j+1}^i) : n-7\le j\le n-6\}$. Then

$$\begin{aligned}
E_{\frac{n-4}{3}-1} &= \cup_{i=1}^{mt}\{|f(v_j^i)-f(v_{j+1}^i)| : n-7\le j\le n-6\}\\
&= \{|f(v_{n-7}^1)-f(v_{n-6}^1)|, |f(v_{n-6}^1)-f(v_{n-5}^1)|\}\\
&\quad \cup\{|f(v_{n-7}^2)-f(v_{n-6}^2)|, |f(v_{n-6}^2)-f(v_{n-5}^2)|\}\\
&\quad \cup,\ldots,\cup\{|f(v_{n-7}^{mt-1})-f(v_{n-6}^{mt-1})|, |f(v_{n-6}^{mt-1})-f(v_{n-5}^{mt-1})|\}\\
&\quad \cup\{|f(v_{n-7}^{mt})-f(v_{n-6}^{mt})|, |f(v_{n-6}^{mt})-f(v_{n-5}^{mt})|\}\\
&= \{|F_{mnt-n+9}-F_{mnt-n+7}|, |F_{mnt-n+7}-F_{mnt-n+5}|\}\\
&\quad \cup\{|F_{mnt-2n+9}-F_{mnt-2n+7}|, |F_{mnt-2n+7}-F_{mnt-2n+5}|\}\\
&\quad \cup,\ldots,\cup\{|F_{n+9}-F_{n+7}|, |F_{n+7}-F_{n+5}|\}\\
&\quad \cup\{|F_9-F_7|, |F_7-F_5|\}\\
&= \{F_{mnt-n+8}, F_{mnt-n+6}\}\cup\{F_{mnt-2n+8}, F_{mnt-2n+6}\}\cup,\ldots,\\
&\quad \cup\{F_{n+8}, F_{n+6}\}\cup\{F_8, F_6\}.
\end{aligned}$$

Similarly, for the edge labeling between the end vertex v_{n-5}^i of the $(\frac{n-4}{3}-1)^{th}$ loop and starting vertex v_{n-4}^i of the $(\frac{n-4}{3})^{rd}$ loop, let $E^1_{\frac{n-4}{3}-1}=\{f^*(v_{n-5}^iv_{n-4}^i) : 1\le i\le mt\}$. Calcula-

tion shows that

$$\begin{aligned}
E^1_{\frac{n-4}{3}-1} &= \{|f(v^i_{n-5}) - f(v^i_{n-4})| : 1 \le i \le mt\} \\
&= \{|f(v^1_{n-5}) - f(v^1_{n-4})|, |f(v^2_{n-5}) - f(v^2_{n-4})|, \ldots, \\
&\quad |f(v^{mt-1}_{n-5}) - f(v^{mt-1}_{n-4})|, |f(v^{mt}_{n-5}) - f(v^{mt}_{n-4})|\} \\
&= \{|F_{mnt-n+5} - F_{mnt-n+6}|, |F_{mnt-2n+5} - F_{mnt-2n+6}|, \ldots, \\
&\quad |F_{n+5} - F_{n+6}|, |F_5 - F_6|\} \\
&= \{F_{mnt-n+4}, F_{mnt-2n+4}, \ldots, F_{n+4}, F_4\}.
\end{aligned}$$

Now for $s = \dfrac{n-4}{3}$, let $E_{\frac{n-4}{3}} = \bigcup\limits_{i=1}^{mt} \{f^*(v^i_j v^i_{j+1}) : n-4 \le j \le n-3\}$. Then

$$\begin{aligned}
E_{\frac{n-4}{3}} &= \cup_{i=1}^{mt} \{|f(v^i_j) - f(v^i_{j+1})| : n-4 \le j \le n-3\} \\
&= \{|f(v^1_{n-4}) - f(v^1_{n-3})|, |f(v^1_{n-3}) - f(v^1_{n-2})|\} \\
&\quad \cup \{|f(v^2_{n-4}) - f(v^2_{n-3})|, |f(v^2_{n-3}) - f(v^2_{n-2})|\} \cup \\
&\quad , \ldots, \{|f(v^{mt-1}_{n-4}) - f(v^{mt-1}_{n-3})|, |f(v^{mt-1}_{n-4}) - f(v^{mt-1}_{n-3})|\} \\
&\quad \cup \{|f(v^{mt}_{n-4}) - f(v^{mt}_{n-3})|, |f(v^{mt}_{n-3}) - f(v^{mt}_{n-2})|\} \\
&= \{|F_{mnt-n+6} - F_{mnt-n+4}|, |F_{mnt-n+4} - F_{mnt-n+2}|\} \\
&\quad \cup \{|F_{mnt-2n+6} - F_{mnt-2n+4}|, |F_{mnt-2n+4} - F_{mnt-2n+2}|\} \\
&\quad \cup, \ldots, \cup \{|F_{n+6} - F_{n+4}|, |F_{n+4} - F_{n+2}|\} \\
&\quad \cup \{|F_6 - F_4|, |F_4 - F_2|\} \\
&= \{F_{mnt-n+5}, F_{mnt-n+3}\} \cup \{F_{mnt-2n+5}, F_{mnt-2n+3}\} \cup, \ldots, \\
&\quad \cup \{F_{n+5}, F_{n+3}\} \cup \{F_5, F_3\}.
\end{aligned}$$

We find the edge labeling between the end vertex v^i_{n-2} of the $(\frac{n-4}{3})^{rd}$ loop and the vertex v^i_{n-1}. Let $E^*_1 = \{f^*(v^i_{n-2} v^i_{n-1}) : 1 \le i \le mt\}$. Then

$$\begin{aligned}
E^*_1 &= \{|f(v^i_{n-2}) - f(v^i_{n-1})| : 1 \le i \le mt\} \\
&= \{|f(v^1_{n-2}) - f(v^1_{n-1})|, |f(v^2_{n-2}) - f(v^2_{n-1})|, \ldots, |f(v^{mt-1}_{n-2}) - f(v^{mt-1}_{n-1})|, \\
&\quad |f(v^{mt}_{n-2}) - f(v^{mt}_{n-1})|\} \\
&= \{|F_{mnt-n+2} - F_{mnt-n+3}|, |F_{mnt-2n+2} - F_{mnt-2n+3}|, \ldots, \\
&\quad |F_{n+2} - F_{n+3}|, |F_2 - F_3|\} \\
&= \{F_{mnt-n+1}, F_{mnt-2n+1}, \ldots, F_{n+1}, F_1\}.
\end{aligned}$$

Let $E^*_2 = \{f^*(v^i_{n-1} v^i_n) : 1 \le i \le mt\}$. Then

$$\begin{aligned}
E^*_2 &= \{|f(v^i_{n-1}) - f(v^i_n)| : 1 \le i \le mt\} \\
&= \{|f(v^1_{n-1}) - f(v^1_n)|, |f(v^2_{n-1}) - f(v^2_n)|, \ldots, \\
&\quad |f(v^{mt-1}_{n-1}) - f(v^{mt-1}_n)|, |f(v^{mt}_{n-1}) - f(v^{mt}_n)|\} \\
&= \{|F_{mnt-n+3} - F_{mnt-n+1}|, |F_{mnt-2n+3} - F_{mnt-2n+1}|, \ldots, \\
&\quad |F_{n+3} - F_{n+1}|, |F_3 - F_1|\} \\
&= \{F_{mnt-n+2}, F_{mnt-2n+2}, \ldots, F_{n+2}, F_2\}.
\end{aligned}$$

Therefore,

$$\begin{aligned} E &= (E_1 \cup E_2 \cup, \ldots, \cup E_{\frac{n-4}{3}}) \cup (E_4^1 \cup E_5^1 \cup, \ldots, \cup E^1_{\frac{n-4}{3}-1}) \cup E_1^* \cup E_2^* \\ &= \{F_1, F_2, \ldots, F_{mnt}\} \end{aligned}$$

Thus, $S^t_{m,n}$ admits a super fibonacci graceful labeling. $\square$

For example the super fibonacci graceful labeling of $S^2_{3,7}$ is shown in Fig.3.

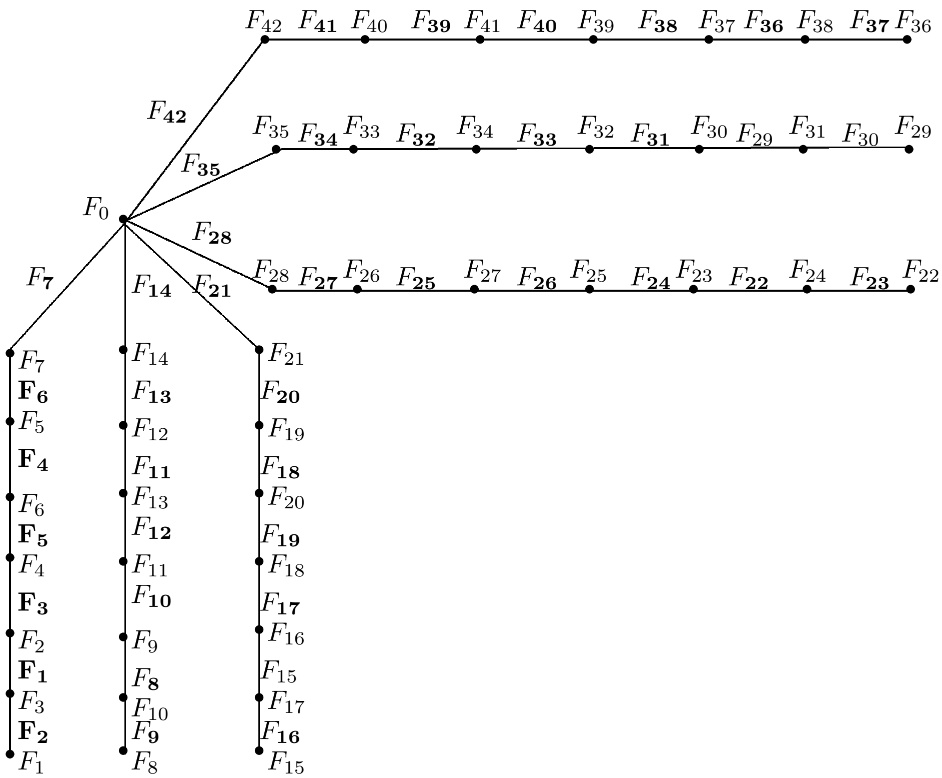

Fig.3

Theorem 2.9 *The complete graph K_n is a super fibonacci graceful graph if $n \leq 3$.*

Proof Let $\{v_0, v_1, \ldots, v_{n-1}\}$ be the vertex set of K_n. Then $v_i (0 \leq i \leq n-1)$ is adjacent to all other vertices of K_n. Let v_0 and v_1 be labeled as F_0 and F_q respectively. Then v_2 must be given F_{q-1} or F_{q-2} so that the edge v_1v_2 will receive a fibonacci number F_{q-2} or F_{q-1}. Therefore, the edges will receive the distinct labeling. Suppose not, Let v_0 and v_1 be labeled as F_1 and F_q or F_0 and F_{q-2} respectively. Then v_2 must be given F_{q-1} or F_{q-2} so that the edges v_0v_2 and v_1v_2 will receive the same edge label F_{q-2}, which is a contradiction by our definition. Hence, K_n is super fibonacci graceful graph if $n \leq 3$. $\square$

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Surface Embeddability of Graphs via Tree-travels

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Abstract: This paper provides a characterization for surface embeddability of a graph with any given orientable and nonorientable genus not zero via a method discovered by the author thirty years ago.

Key Words: Surface, graph,Smarandache λ^S-drawing, embeddability, tree-travel.

AMS(2010): 05C15, 05C25

§1. Introduction

A drawing of a graph G on a surface S is such a drawing with no edge crosses itself, no adjacent edges cross each other, no two edges intersect more than once, and no three edges have a common point. A *Smarandache λ^S-drawing* of G on S is a drawing of G on S with minimal intersections λ^S. Particularly, a Smarandache 0-drawing of G on S, if existing, is called an embedding of G on S. Along the Kurotowski research line for determining the embeddability of a graph on a surface of genus not zero, the number of forbidden minors is greater than a hundred even for the projective plane, a nonorientable surface of genus 1 in [1].

However, this paper extends the results in [3] which is on the basis of the method established in [3-4] by the author himself for dealing with the problem on the maximum genus of a graph in 1979. Although the principle idea looks like from the joint trees, a main difference of a tree used here is not corresponding to an embedding of the graph considered.

Given a graph $G = (V, E)$, let T be a spanning tree of G. If each cotree edge is added to T as an articulate edge, what obtained is called a *protracted tree* of G, denoted by $\breve{T}$. An protracted tree $\breve{T}$ is oriented via an orientation of T or its fundamental circuits. In order to guarantee the well-definedness of the orientation for given rotation at all vertices on G and a selected vertex of T, the direction of a cotree edge is always chosen in coincidence with its direction firstly appeared along the the face boundary of $\breve{T}$. For convenience, vertices on the boundary are marked by the ordinary natural numbers as the root vertex, the starting vertex, by 0. Of course, the boundary is a travel on G, called a *tree-travel*.

In Fig.1, (a) A spanning tree T of K_5(i.e., the complete graph of order 5), as shown by bold lines; (b) the protracted tree $\breve{T}$ of T.

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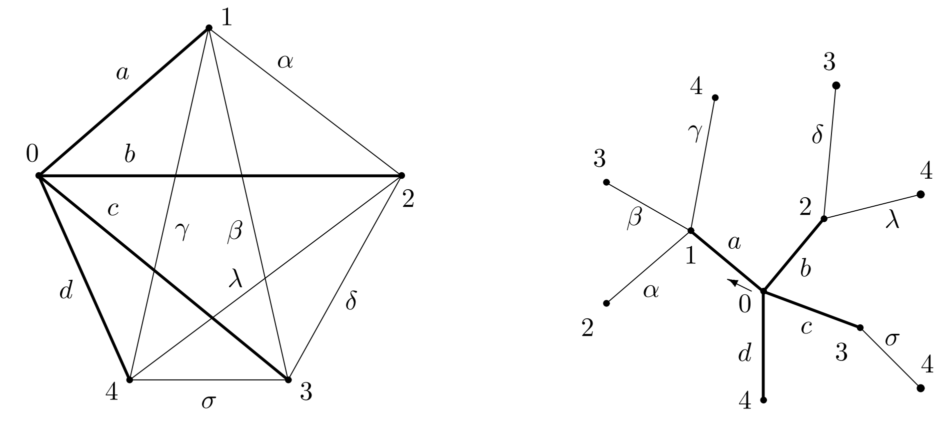


(a) T (b) $\breve{T}$

Fig.1

§2. Tree-Travels

Let $C = C(V; e)$ be the tree travel obtained from the boundary of $\breve{T}$ with 0 as the starting vertex. Apparently, the travel as a edge sequence $C = C(e)$ provides a double covering of $G = (V, E)$, denoted by

$$C(V; e) = 0P_{0,i_1}v_{i_1}P_{i_1,i_2}v_{i_2}P_{i_2,i'_1}v_{i'_1}P_{i'_1,i'_2}v_{i'_2}P_{i'_2,2\epsilon}0 \tag{1}$$

where $\epsilon = |E|$.

For a vertex-edge sequence Q as a tree-travel, denote by $[Q]_{\rm eg}$ the edge sequence induced from Q missing vertices, then $C_{\rm eg} = [C(V; e)]_{\rm eg}$ is a polyhegon(*i.e.*, a polyhedron with only one face).

Example 1 From $\breve{T}$ in Fig.1(b), obtain the thee-travel

$$C(V; e) = 0P_{0,8}0P_{8,14}0P_{14,18}0P_{18,20}0$$

where $v_0 = v_8 = v_{14} = v_{18} = v_{20} = 0$ and

$$P_{0,8} = a1\alpha 2\alpha^{-1}1\beta 3\beta^{-1}1\gamma 4\gamma^{-1}1a^{-1};$$
$$P_{8,14} = b2\delta 3\delta^{-1}2\lambda 4\lambda^{-1}2b^{-1};$$
$$P_{14,18} = c3\sigma 4\sigma^{-1}3c^{-1};$$
$$P_{18,20} = d4d^{-1}.$$

For natural number i, if av_ia^{-1} is a segment in C, then a is called a *reflective edge* and then v_i, the *reflective vertex* of a.

Because of nothing important for articulate vertices(1-valent vertices) and 2-valent vertices in an embedding, we are allowed to restrict ourselves only discussing graphs with neither 1-valent nor 2-valent vertices without loss of generality. From vertices of all greater than or equal to 3, we are allowed only to consider all reflective edges as on the cotree.

If v_{i_1} and v_{i_2} are both reflective vertices in (1), their reflective edges are adjacent in G and $v_{i'_1} = v_{i_1}$ and $v_{i'_2} = v_{i_2}$, $[P_{v_{i_1,i_2}}]_{\rm eg} \cap [P_{v_{i'_1,i'_2}}]_{\rm eg} = \emptyset$, but neither $v_{i'_1}$ nor $v_{i'_2}$ is a reflective vertex, then the transformation from C to

$$\triangle_{v_{i_1},v_{i_2}} C(V;e) = 0P_{0,i_1} v_{i_1} P_{i'_1,i'_2} v_{i_2} P_{i_2,i'_1} v_{i'_1} P_{i_1,i_2} v_{i'_2} P_{i'_2,0} 0. \tag{2}$$

is called an operation of *interchange segments* for $\{v_{i_1}, v_{i_2}\}$.

Example 2 In $C = C(V;e))$ of Example 1, $v_2 = 2$ and $v_4 = 3$ are two reflective vertices, their reflective edges α and β, $v_9 = 2$ and $v_15 = 3$. For interchange segments once on C, we have

$$\triangle_{2,3} C = 0P_{0,2} 2P_{9,15} 3P_{4,9} 2P_{2,4} 3P_{15,20} 0 \ (= C_1).$$

where

$$\begin{aligned}
&P_{0,2} = a1\alpha \ (= P_{1;0,2});\\
&P_{9,15} = \delta 3\delta^{-1} 2\lambda 4\lambda^{-1} 2b^{-1} 0c3 \ (= P_{1;2,8});\\
&P_{4,9} = \beta^{-1} 1\gamma 4\gamma^{-1} 1a^{-1} 0b2 \ (= P_{1;8,13});\\
&P_{2,4} = \alpha^{-1} 1\beta \ (= P_{1;13,15});\\
&P_{15,20} = \sigma 4\sigma^{-1} 3c^{-1} 0d4d^{-1} \ (= P_{1;15,20}).
\end{aligned}$$

Lemma 1 Polyhegon $\triangle_{v_i,v_j} C_{\rm eg}$ is orientable if, and only if, $C_{\rm eg}$ is orientable and the genus of $\triangle_{v_{i_1},v_{i_2}} C_{\rm eg}$ is exactly 1 greater than that of $C_{\rm eg}$.

Proof Because of the invariant of orientability for $\triangle$-operation on a polyhegon, the first statement is true.

In order to prove the second statement, assume cotree edges α and β are reflective edges at vertices, respectively,v_{i_1} and v_{i_2}. Because of

$$C_{\rm eg} = A\alpha\alpha^{-1} B\beta\beta^{-1} CDE$$

where

$$\begin{aligned}
&A\alpha = [P_{0,i_1}]_{\rm eg};\ \alpha^{-1}B\beta = [P_{i_1,i_2}]_{\rm eg};\\
&\beta^{-1}C = [P_{i_2,i'_1}]_{\rm eg};\ D = [P_{i'_1,i'_2}]_{\rm eg};\\
&E = [P_{i'_2,i_\epsilon}]_{\rm eg},
\end{aligned}$$

we have

$$\begin{aligned}
\triangle_{v_{i_1},v_{i_2}} C_{\rm eg} &= A\alpha D\beta^{-1} C\alpha^{-1} B\beta E\\
&\sim_{\rm top} ABCDE\alpha\beta\alpha^{-1}\beta^{-1},\ \text{(Theorem 3.3.3 in [5])}\\
&= C_{\rm eg}\alpha\beta\alpha^{-1}\beta^{-1}\ \text{(Transform 1, in §3.1 of [5])}.
\end{aligned}$$

Therefore, the second statement is true. □

If interchange segments can be done on C successively for k times, then C is called a *k-tree travel*. Since one reflective edge is reduced for each interchange of segments on C and C has at most $m = \lfloor \beta/2 \rfloor$ reflective edges, we have $0 \leqslant k \leqslant m$ where $\beta = \beta(G)$ is the Betti number(or corank) of G. When $k = m$, C is also called *normal*.

For a k-tree travel $C_k(V; e, e^{-1})$ of G, graph G_k is defined as

$$G_k = T \bigcup [E_{\rm ref} \bigcap E_{\bar{T}} - \sum_{j=1}^{k} \{e_j, e'_j\}] \tag{3}$$

where T is a spanning tree, $[X]$ represents the edge induced subgraph by edge subset X, and $e \in E_{\rm ref}$, $e \in E_{\bar{T}}$, $\{e_j, e'_j\}$ are, respectively, reflective edge, cotree edge, pair of reflective edges for interchange segments.

Example 3 On C_1 in Example 2, $v_{1;3} = 3$ and $v_{1;5} = 4$ are two reflective vertices, $v_{1;8} = 3$ and $v_{1;10} = 4$. By doing interchange segments on C_1, obtain

$$\triangle_{3,4} C_1 = 0P_{1;0,10}3P_{1;17,19}4P_{1;12,15}3P_{1;10.12}4P_{1;19,20}0 \ (= C_2)$$

where

$$P_{1;0,10} = a1\alpha 2b^{-1}0c3\beta^{-1}1\gamma 4\gamma^{-1}1a^{-1}0b2\delta(= P_{2;0,10});$$
$$P_{1;17,19} = c^{-1}0d(= P_{2;10,12});$$

$P_{1;12.17} = \alpha^{-1}2\alpha^{-1}1\beta 3\sigma 4\sigma^{-1}(= P_{2;12,17});$

$$P_{1;10,12} = \delta^{-1}2\lambda(= P_{2;17,19});$$
$$P_{1;19,20} = d^{-1}(= P_{2;19,20}).$$

Because of $[P_{2;6,16}]_{\rm eg} \cap [P_{2;12,19}]_{\rm eg} \neq \emptyset$ for $v_{2;12} = 4$ and $v_{2;19} = 4$, only $v_{2;6} = 4$ and $v_{2;16} = 4$ with their reflective edges γ and σ are allowed for doing interchange segments on C_2. The protracted tree $\breve{T}$ in Fig.1(b) provides a 2-tree travel C, and then a 1-tree travel as well.

However, if interchange segments are done for pairs of cotree edges as $\{\beta, \gamma\}$, $\{\delta, \lambda\}$ and $\{\alpha, \sigma\}$ in this order, it is known that C is also a 3-tree travel.

On C of Example 1, the reflective vertices of cotree edges β and γ are, respectively, $v_4 = 3$ and $v_6 = 4$, choose $4' = 15$ and $6' = 19$, we have

$$\triangle_{4,6} C = 0P_{1;0,4}3P_{1;4,8}4P_{1;8,17}3P_{1;17,19}4P_{1;19,20}0(= C_1)$$

where

$$P_{1;0,4} = P_{0,4};\ P_{1;4,8} = P_{15,19};\ P_{1;8,17} = P_{6,15};$$
$$P_{1;17,19} = P_{4,6};\ P_{1;19,20} = P_{19,20}.$$

On C_1, subindices of the reflective vertices for reflective edges δ and λ are 5 and 8, choose $5' = 17$ and $8' = 19$, find

$$\triangle_{5,8} C_1 = 0P_{2;0,5}3P_{2;5,7}4P_{2;7,16}3P_{2;16,19}4P_{2;19,20}0(= C_2)$$

where

$$P_{2;0,12} = P_{1;0,12};\ P_{2;12,14} = P_{1;17,19};\ P_{2;14,17} = P_{1;14,17};$$
$$P_{2;17,19} = P_{1;12,14};\ P_{2;19,20} = P_{1;19,20}.$$

On C_2, subindices of the reflective vertices for reflective edges α and σ are 2 and 5, choose $2' = 18$ and $5' = 19$, find

$$\triangle_{5,8}C_2 = 0P_{3;0,2}3P_{3;2,3}4P_{3;3,16}3P_{3;16,19}4P_{3;19,20}0(= C_3)$$

where

$$P_{3;0,2} = P_{2;0,2};\ P_{2;2,3} = P_{2;18,19};\ P_{3;3,16} = P_{2;5,18};$$
$$P_{3;16,19} = P_{2;2,5};\ P_{3;19,20} = P_{2;19,20}.$$

Because of $\beta(K_5) = 6$, $m = 3 = \lfloor \beta/2 \rfloor$. Thus, the tree-travel C is normal.

This example tells us the problem of determining the maximum orientable genus of a graph can be transformed into that of determining a k-tree travel of a graph with k maximum as shown in [4].

Lemma 2 Among all k-tree travel of a graph G, the maximum of k is the maximum orientable genus $\gamma_{\max}(G)$ of G.

Proof In order to prove this lemma, the following two facts have to be known(both of them can be done via the finite recursion principle in §1.3 of [5]!).

Fact 1 In a connected graph G considered, there exists a spanning tree such that any pair of cotree edges whose fundamental circuits with vertex in common are adjacent in G.

Fact 2 For a spanning tree T with Fact 1, there exists an orientation such that on the protracted tree $\breve{T}$, no two articulate subvertices(articulate vertices of T) with odd out-degree of cotree have a path in the cotree.

Because of that if two cotree edges for a tree are with their fundamental circuits without vertex in common then they for any other tree are with their fundamental circuits without vertex in common as well, Fact 1 enables us to find a spanning tree with number of pairs of adjacent cotree edges as much as possible and Fact 2 enables us to find an orientation such that the number of times for dong interchange segments successively as much as possible. From Lemma 1, the lemma can be done. □

§3. Tree-Travel Theorems

The purpose of what follows is for characterizing the embeddability of a graph on a surface of genus not necessary to be zero via k-tree travels.

Theorem 1 *A graph G can be embedded into an orientable surface of genus k if, and only if, there exists a k-tree travel $C_k(V;e)$ such that G_k is planar.*

Proof Necessity. Let $\mu(G)$ be an embedding of G on an orientable surface of genus k. From Lemma 2, $\mu(G)$ has a spanning tree T with its edge subsets E_0, $|E_0| = \beta(G) - 2k$, such that $\hat{G} = G - E_0$ is with exactly one face. By successively doing the inverse of interchange segments for k times, a k-tree travel is obtained on $\hat{G}$. Let K be consisted of the k pairs of cotree edge subsets. Thus, from Operation 2 in §3.3 of [5], $G_k = G - K = \hat{G} - K + E_0$ is planar.

Sufficiency. Because of G with a k-tree travel $C_k(V; e)$, Let K be consisted of the k pairs of cotree edge subsets in successively doing interchange segments for k times. Since $G_k = G - K$ is planar, By successively doing the inverse of interchange segments for k times on $C_k(V; e)$ in its planar embedding, an embedding of G on an orientable surface of genus k is obtained. □

Example 4 In Example 1, for $G = K_5$, C is a 1-tree travel for the pair of cotree edges α and β. And, $G_1 = K_5 - \{\alpha, \beta\}$ is planar. Its planar embedding is

$$\begin{aligned}
&[4\sigma^{-1}3c^{-1}0d4]_{\mathrm{eg}} = (\sigma^{-1}c^{-1}d);\\
&[4d^{-1}0a1\gamma 4]_{\mathrm{eg}} = (d^{-1}a\gamma);\\
&[3\sigma 4\lambda^{-1}2\delta 3]_{\mathrm{eg}} = (\sigma\lambda^{-1}\delta);\ [0c3\delta^{-1}2b^{-1}0]_{\mathrm{eg}} = (c\delta^{-1}b^{-1});\\
&[2\lambda 4\gamma^{-1}1a^{-1}0b2]_{\mathrm{eg}} = (\lambda\gamma^{-1}a^{-1}b).
\end{aligned}$$

By recovering $\{\alpha, \beta\}$ to G and then doing interchange segments once on C, obtain C_1. From C_1 on the basis of a planar embedding of G_1, an embedding of G on an orientable surface of genus 1(the torus) is produced as

$$\begin{aligned}
&[4\sigma^{-1}3c^{-1}0d4]_{\mathrm{eg}} = (\sigma^{-1}c^{-1}d);\ [4d^{-1}0a1\gamma 4]_{\mathrm{eg}} = (d^{-1}a\gamma);\\
&[3\sigma 4\lambda^{-1}2\delta 3\beta^{-1}1a^{-1}0b2\alpha^{-1}1\beta 3]_{\mathrm{eg}} = (\sigma\lambda^{-1}\delta\beta^{-1}a^{-1}b2\alpha^{-1}\beta);\\
&[0c3\delta^{-1}2b^{-1}0]_{\mathrm{eg}} = (c\delta^{-1}b^{-1});\ [2\lambda 4\gamma^{-1}1\alpha 2]_{\mathrm{eg}} = (\lambda\gamma^{-1}\alpha).
\end{aligned}$$

Similarly, we further discuss on nonorientable case. Let $G = (V, E)$, T a spanning tree, and

$$C(V; e) = 0P_{0,i}v_iP_{i,j}v_jP_{j,2\epsilon}0 \tag{4}$$

is the travel obtained from 0 along the boundary of protracted tree $\breve{T}$. If v_i is a reflective vertex and $v_j = v_i$, then

$$\widetilde{\triangle}_\xi C(V; e) = 0P_{0,i}v_iP_{i,j}^{-1}v_jP_{j,2\epsilon}0 \tag{5}$$

is called what is obtained by doing a *reverse segment* for the reflective vertex v_i on $C(V; e)$.

If reverse segment can be done for successively k times on C, then C is called a $\widetilde{k}$*-tree travel*. Because of one reflective edge reduced for each reverse segment and at most β reflective edges on C, we have $0 \leqslant k \leqslant \beta$ where $\beta = \beta(G)$ is the Betti number of G(or corank). When $k = \beta$, C(or G) is called *twist normal*.

Lemma 3 A connected graph is twist normal if, and only if, the graph is not a tree.

Proof Because of trees no cotree edge themselves, the reverse segment can not be done, this leads to the necessity. Conversely, because of a graph not a tree, the graph has to be with a circuit, a tree-travel has at least one reflective edge. Because of no effect to other reflective

edges after doing reverse segment once for a reflective edge, reverse segment can always be done for successively $\beta = \beta(G)$ times, and hence this tree-travel is twist normal. Therefore, sufficiency holds. $\square$

Lemma 4 Let C be obtained by doing reverse segment at least once on a tree-travel of a graph. Then the polyhegon $[\triangle_i C]_{\text{eg}}$ is nonorientable and its genus

$$\widetilde{g}([\triangle_\xi C]_{\text{eg}}) = \begin{cases} 2g(C)+1, & \text{when } C \text{ orientable;} \\ \widetilde{g}(C)+1, & \text{when } C \text{ nonorientable.} \end{cases} \tag{6}$$

Proof Although a tree-travel is orientable with genus 0 itself, after the first time of doing the reverse segment on what are obtained the nonorientability is always kept unchanged. This leads to the first conclusion. Assume C_{eg} is orientable with genus $g(C)$(in fact, only $g(C) = 0$ will be used!). Because of

$$[\triangle_i C]_{\text{eg}} = A\xi B^{-1}\xi C$$

where $[P_{0,i}]_{\text{eg}} = A\xi$, $[P_{i,j}]_{\text{eg}} = \xi^{-1}B$ and $[P_{j,\epsilon}]_{\text{eg}} = C$, From (3.1.2) in [5]

$$[\triangle_i C]_{\text{eg}} \sim_{\text{top}} ABC\xi\xi.$$

Noticing that from Operation 0 in §3.3 of [5], $C_{rseg} \sim_{\text{top}} ABC$, Lemma 3.1.1 in [5] leads to

$$\widetilde{g}([\triangle_\xi C]_{\text{eg}}) = 2g([C]_{\text{eg}}) + 1 = 2g(C) + 1.$$

Assume C_{eg} is nonorientable with genus $g(C)$. Because of

$$C_{\text{eg}} = A\xi\xi^{-1}BC \sim_{\text{top}} ABC,$$

$\widetilde{g}([\triangle_\xi C]_{\text{eg}}) = \widetilde{g}(C) + 1$. Thus, this implies the second conclusion. $\square$

As a matter of fact, only reverse segment is enough on a tree-travel for determining the nonorientable maximum genus of a graph.

Lemma 5 Any connected graph, except only for trees, has its Betti number as the nonorientable maximum genus.

Proof From Lemmas 3-4, the conclusion can soon be done. $\square$

For a $\widetilde{k}$-tree travel $C_{\widetilde{k}}(V; e)$ on G, the graph $G_{\widetilde{k}}$ is defined as

$$G_{\widetilde{k}} = T \bigcup [E_{\text{ref}} - \sum_{j=1}^{k} \{e_j\}] \tag{7}$$

where T is a spanning tree, $[X]$ the induced graph of edge subset X, and $e \in E_{\text{ref}}$ and $\{e_j, e'_j\}$, respectively, a reflective edge and that used for reverse segment.

Theorem 2 *A graph G can be embedded into a nonorientable surface of genus k if, and only if, G has a $\widetilde{k}$-tree travel $C_{\widetilde{k}}(V; e)$ such that $G_{\widetilde{k}}$ is planar.*

Proof From Lemma 3, for k, $1 \leqslant k \leqslant \beta(G)$, any connected graph G but tree has a $\widetilde{k}$-tree travel.

Necessity. Because of G embeddable on a nonorientable surface $S_{\widetilde{k}}$ of genus k, let $\widetilde{\mu}(G)$ be an embedding of G on $S_{\widetilde{k}}$. From Lemma 5, $\widetilde{\mu}(G)$ has a spanning tree T with cotree edge set E_0, $|E_0| = \beta(G) - k$, such that $\widetilde{G} = G - E_0$ has exactly one face. By doing the inverse of reverse segment for k times, a $\widetilde{k}$-tree travel of $\widetilde{G}$ is obtained. Let K be a set consisted of the k cotree edges. From Operation 2 in §3.3 of [5], $G_{\widetilde{k}} = G - K = \widetilde{G} - K + E_0$ is planar.

Sufficiency. Because of G with a $\widetilde{k}$-tree travel $C_{\widetilde{k}}(V;e)$, let K be the set of k cotree edges used for successively dong reverse segment. Since $G_{\widetilde{k}} = G - K$ is planar, by successively doing reverse segment for k times on $C_{\widetilde{k}}(V;e)$ in a planar embedding of $G_{\widetilde{k}}$, an embedding of G on a nonorientable surface $S_{\widetilde{k}}$ of genus k is then extracted. □

Example 5 On $K_{3,3}$, take a spanning tree T, as shown in Fig.2(a) by bold lines. In (b), given a protracted tree $\breve{T}$ of T. From $\breve{T}$, get a tree-travel

$$C = 0P_{0,11}2P_{11,15}2P_{15,0}0 \ (= C_0)$$

where $v_0 = v_{18}$ and

$$\begin{aligned}
&P_{0,11} = c4\delta 5\delta^{-1}4\gamma 3\gamma^{-1}4c^{-1}0d2e3\beta 1\beta^{-1}3e^{-1};\\
&P_{11,15} = d^{-1}0a1b5\alpha;\\
&P_{15,0} = \alpha^{-1}5b^{-1}1a^{-1}.
\end{aligned}$$

Because of $v_{15} = 2$ as the reflective vertex of cotree edge α and $v_{11} = v_{15}$,

$$\triangle_3 C_0 = 0P_{1;0,11}2P_{1;11,15}2P_{1;15,0}0 \ (= C_1)$$

where

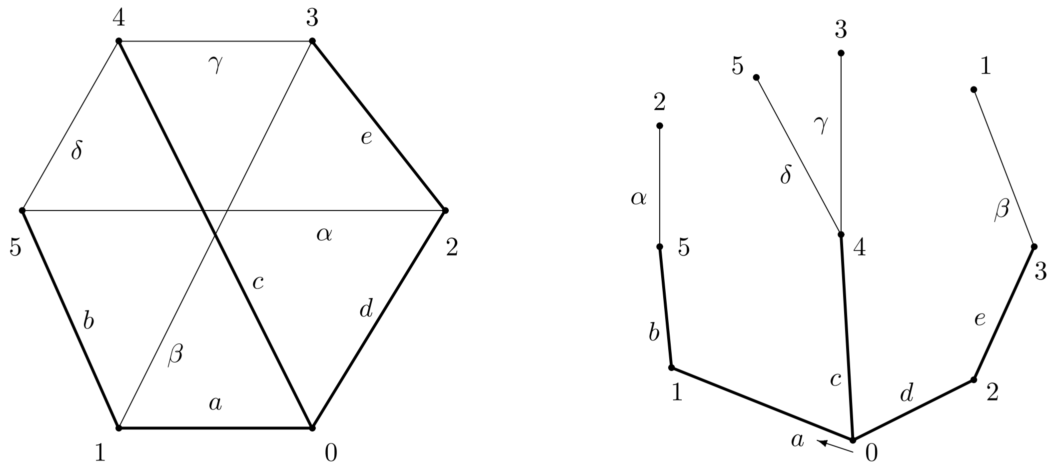
$$\begin{aligned}
&P_{1;0,11} = P_{0,11} = c4\delta 5\delta^{-1}4\gamma 3\gamma^{-1}4c^{-1}0d2e3\beta 1\beta^{-1}3e^{-1};\\
&P_{1;11,15} = P_{11,15}^{-1} = \alpha^{-1}5b^{-1}1a^{-1}0d;\\
&P_{1;15,0} = P_{15,0} = \alpha^{-1}5b^{-1}1a^{-1}.
\end{aligned}$$

Since $G_{\widetilde{1}} = K_{3,3} - \alpha$ is planar, from C_0 we have its planar embedding

$$\begin{aligned}
&f_1 = [5P_{16,0}0P_{0,2}0]_{\text{eg}} = (b^{-1}a^{-1}c\delta);\\
&f_2 = [3P_{4,8}3]_{\text{eg}} = (\gamma^{-1}c^{-1}de);\\
&f_3 = [1P_{13,14}5P_{2,4}3P_{8,9}1]_{\text{eg}} = (\delta^{-1}\gamma\beta b);\\
&f_4 = [1P_{9,13}1]_{\text{eg}} = (\beta^{-1}e^{-1}d^{-1}a).
\end{aligned}$$

By doing reverse segment on C_0, get C_1. On this basis, an embedding of $K_{3,3}$ on the projective plane(*i.e.*, nonorientable surface $S_{\widetilde{1}}$ of genus 1) is obtained as

$$\begin{cases}
\widetilde{f_1} = [5P_{1;16,0}0P_{1;0,2}0]_{\text{eg}} = f_1 = (b^{-1}a^{-1}c\delta);\\
\widetilde{f_2} = [3P_{1;4,8}3]_{\text{eg}} = f_2 = (\gamma^{-1}c^{-1}de);\\
\widetilde{f_3} = [1P_{1;9,11}2P_{1;11,13}1]_{\text{eg}} = be^{-1}e^{-1}\alpha^{-1}b^{-1});\\
\widetilde{f_4} = [0P_{1;14,15}2P_{1;15,16}5P_{1;2,4}3P_{1;8,9}1P_{1;13,14}0]_{\text{eg}}\\
\quad = (d\alpha^{-1}\delta^{-1}\gamma\beta a^{-1}).
\end{cases}$$



(a) T (b) $\check{T}$

Fig.2

§4. Research Notes

A. For the embeddability of a graph on the torus, double torus *etc* or in general orientable surfaces of genus small, more efficient characterizations are still necessary to be further contemplated on the basis of Theorem 1.

B. For the embeddability of a graph on the projective plane(1-crosscap), Klein bottle(2-crosscap), 3-crosscap *etc* or in general nonorientable surfaces of genus small, more efficient characterizations are also necessary to be further contemplated on the basis of Theorem 2.

C. Tree-travels can be extended to deal with all problems related to embedings of a graph on surfaces as joint trees in a constructive way.

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Edge Maximal C_3 and C_5-Edge Disjoint Free Graphs

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Abstract: For a positive integer k, let $\mathcal{G}(n;E_{2k+1})$ be the class of graphs on n vertices containing no two $2k+1$-edge disjoint cycles. Let $f(n;E_{2k+1}) = \max\{\mathcal{E}(G) : G \in \mathcal{G}(n;E_{2k+1})\}$. In this paper we determine $f(n;E_{2k+1})$ and characterize the edge maximal members in $\mathcal{G}(n;E_{2k+1})$ for $k = 1$ and 2.

Key Words: Extremal graphs; Edge disjoint: Cycles, Smarandache-Turán graph.

AMS(2010):

§1. Introduction

For our purposes a graph G is finite, undirected and has no loops or multiple edges. We denote the vertex set of G by $V(G)$ and edge set of G by $E(G)$. The cardinalities of these sets are denoted by $\nu(G)$ and $\mathcal{E}(G)$, respectively. The cycle on n vertices is denoted by C_n. Let G be a graph and $u \in V(G)$. The degree of u in G, denoted by $d_G(u)$, is the number of edges of G incident to u. The neighbor set of u in G is a subgraph H of G, denoted by $N_H(u)$, consists of the vertices of H adjacent to u; observe that $d_G(u) = |N_H(u)|$. For a proper subgraph H of G we write $G[V(H)]$ and G-$V(H)$ simply as $G[H]$ and $G - H$ respectively.

Let G_1 and G_2 be graphs. The union $G_1 \cup G_2$ of G_1 and G_2 is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. G_1 and G_2 are vertex disjoint if and only if $V(G_1) \cap V(G_2) = \varnothing$; G_1 and G_2 are edge disjoint if $E(G_1) \cap E(G_2) = \varnothing$. If G_1 and G_2 are vertex disjoint, we denote their union by $G_1 + G_2$. The intersection $G_1 \cap G_2$ of graphs G_1 and G_2 is defined similarly, but in this case we need to assume that $V(G_1) \cap V(G_2) \neq \varnothing$. The join $G \vee H$ of two disjoint graphs G and H is the graph obtained from $G_1 + G_2$ by joining each vertex of G to each vertex of H. For a vertex disjoint subgraphs H_1 and H_2 of G we let $E(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$ and $\mathcal{E}(H_1, H_2) = |E(H_1, H_2)|$.

Let $\mathcal{F}_1, \mathcal{F}_2$ be two graph families and n be a positive integer. Let $\mathcal{G}(n; \mathcal{F}_1, \mathcal{F}_2)$ be a Smarandache-Turán graph family consisting of graphs being $\mathcal{F}_1$-free but containing a subgraph isomorphic to a graph in $\mathcal{F}_2$ on n vertices. Define

$$f(n; \mathcal{F}_1, \mathcal{F}_2) = \max\{|E(G)| : G \in \mathcal{G}(n; \mathcal{F}_1, \mathcal{F}_2)\}.$$

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The problem of determining $f(n;\mathcal{F}_1,\mathcal{F}_2)$ is called the Smarandache-Turán-type extremal problem. It is well known that in case $\mathcal{F}_2=\emptyset$ or $\mathcal{F}_2=\{\text{edge}\}$ the problem is called the Turán-type extremal problem and abbreviated by $f(n;\mathcal{F}_1)$ and the class by $\mathcal{G}(n;\mathcal{F}_1)$. In this paper we consider the Turán-type extremal problem with the odd edge disjoint cycles being the forbidden subgraph. Since a bipartite graph contains no odd cycles, we only consider non-bipartite graphs. For convenience, in the case when $\mathcal{F}_1$ consists of only one member C_r, where r is an odd integer, we write

$$\mathcal{G}(n;r)=\mathcal{G}(n;\mathcal{F}_1), \quad f(n;r) = f(n;\mathcal{F}_1).$$

An important problem in extremal graph theory is that of determining the values of the function $f(n;\mathcal{F}_1)$. Further, characterize the extremal graphs $\mathcal{G}(n;\mathcal{F}_1)$ where $f(n;\mathcal{F}_1)$ is attained. For a given r, the edge maximal graphs of $\mathcal{G}(n;r)$ have been studied by a number of authors [1, 2, 3, 6, 7, 8, 9, 11].

Let $\mathcal{G}(n;E_{2k+1})$ denote the class of graphs on n vertices containing no two $(2k+1)$- edge disjoint cycles. Let

$$f(n;E_{2k+1})=\max\{\mathcal{E}(G):G\in\mathcal{G}(n;E_{2k+1})\}.$$

In this paper we determine $f(n;E_{2k+1})$ and characterize the edge maximal members in for $k=1$ and 2. Now, we state a number of results, which we use to prove our main results.

Lemma 1.1 (Bondy and Murty, [4]) *Let G be a graph on n vertices. If $\mathcal{E}(G)>n^2/4$, then G contains a cycle of length r for each $3\le r\le\lfloor(n+3)/2\rfloor$.*

Theorem 1.2 (Brandt, [5]) *Let G be a non-bipartite graph with n vertices and more than $\lfloor(n-1)^2/4+1\rfloor$ edges. Then G contains all cycles of length between 3 and the length of the longest cycle.*

Let $\mathcal{G}^*(n)$ denote the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\lfloor(n-2)/2\rfloor,\lceil(n-2)/2\rceil}$. For an example of $\mathcal{G}^*(n)$, see Figure 1.

Theorem 1.3 (Jia, [10]) *Let $G\in\mathcal{G}(n;5)$, $n\ge 10$. Then*

$$\mathcal{E}(G)\le\lfloor(n-2)^2/4\rfloor+3.$$

Furthermore, equality holds if and only if $G\in\mathcal{G}^*(n)$.

In this paper we determine $f(n,E_{2k+1})$ and characterize the edge maximal members in $\mathcal{G}(n,E_{2k+1})$ for $k=1,2$, which is the first step toward solving the problem for each positive integer k.

§2. Edge-Maximal C_3-Disjoint Free Graphs

In this section we determine $f(n,E_3)$ and characterize the edge maximal members in $\mathcal{G}(n,E_3)$. We begin with some constructions. Let $\Omega(G)$ denote to the class of graphs obtained by adding

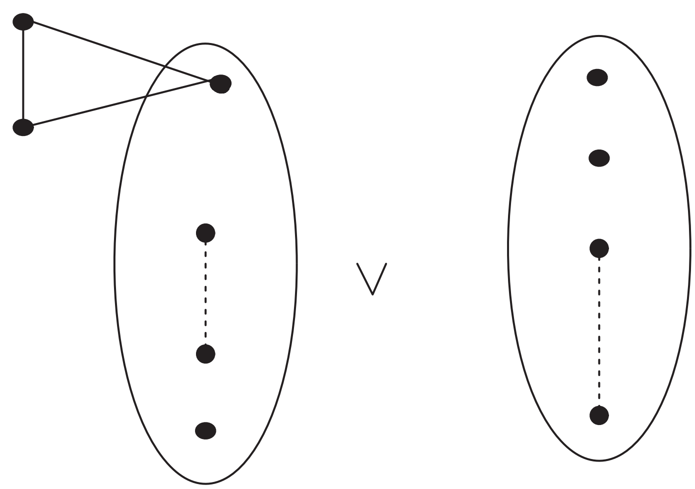

Figure 1: The figure represent a member of $\mathcal{G}^*(n)$.

an edge to the complete bipartite graph $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$. Figure 2 displays a member of $\Omega(G)$. Observe that $\Omega(G) \subseteq \mathcal{G}(n, E_{2k+1})$ and every graph in $\Omega(G)$ contains $\lfloor n^2/4\rfloor + 1$ edges. Thus, we have established that

$$f(n; E_{2k+1}) \geq \lfloor n^2/4\rfloor + 1 \tag{1}$$

We now establish that equality (1) holds for $k = 1$. Further we characterize the edge maximal members in $\mathcal{G}(n; E_3)$.

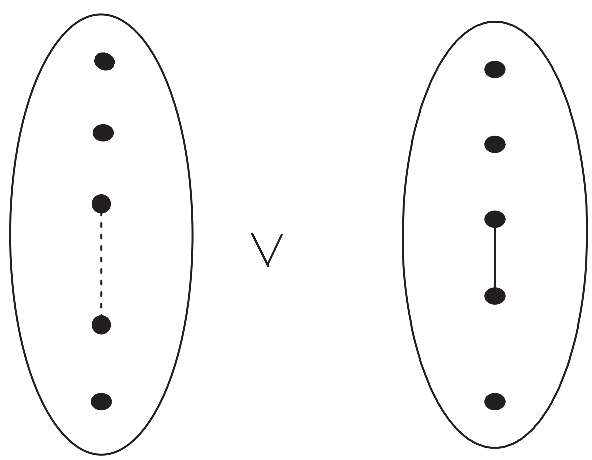

Figure 2: The figure represents a member of $\Omega(G)$.

In the following theorem we determine edge maximum members in $\mathcal{G}(n; E_3)$.

Theorem 2.1 *Let $G \in \mathcal{G}(n; E_3)$. For $n \geq 8$,*

$$f(n; E_3) \leq \lfloor n^2/4 \rfloor + 1.$$

Furthermore, equality holds if and only if $G \in \Omega(G)$.

Proof Let $G \in \mathcal{G}(n, E_3)$. If G contains no cycle of length three, then by Lemma 1.1, $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor$. Thus, $\mathcal{E}(G) < \lfloor n^2/4 \rfloor + 1$. So, we need to consider the case when G has cycles of length 3. Let $xyzx$ be a cycle of length 3 in G. Let $H = G - \{xy, xz, yz\}$. Observe that H cannot have cycles of length 3 as otherwise G would have two edge disjoint cycles of length 3. To this end we consider the following two cases.

Case1: H is not a bipartite graph. Since H contains no cycles of length 3, by Theorem 1.2,

$$\mathcal{E}(G) \leq \lfloor (n-1)^2/4 \rfloor + 1.$$

Now,

$$\begin{aligned} \mathcal{E}(H) &= \mathcal{E}(H) + 3 \\ &\leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 4 < \left\lfloor \frac{n^2}{4} \right\rfloor + 1, \end{aligned}$$

for $n \geq 8$.

Case 2: H is a bipartite graph. Let X and Y be the bipartition of $V(H)$. Thus, $\mathcal{E}(H) \leq |X||Y|$. Observe $|X| + |Y| = n$. The maximum of the above is when $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$. Thus, $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor$. Now we divide our work into two subcases.

Subcase 2.1. All the edges xy, xz, and yz are edges in on of X and Y, say in X. Observe that for any two vertices of Y, say u and v, we have that $\mathcal{E}(\{x, y, z\}, \{u, v\}) \leq 5$ as otherwise G would have two edge disjoint cycles of length 3. Thus, $\mathcal{E}(\{x, y, z\}, Y) \leq 2|Y| + 1$. Now,

$$\begin{aligned} \mathcal{E}(G) &= \mathcal{E}(X - \{x, y, z\}, Y) + \mathcal{E}(\{x, y, z\}, Y) + \mathcal{E}(\{x, y, z\}) \\ &\leq (|X| - 3)|Y| + 2|Y| + 1 + 3 \\ &\leq |X||Y| - |Y| + 4 \leq (|X| - 1)|Y| + 4. \end{aligned}$$

Observe $|X| + |Y| = n$. The maximum of the above equation is when $|X| = \lfloor (n+1)/2 \rfloor$ and $|Y| = \lceil (n-1)/2 \rceil$, Thus,

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 4 < \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

Subcase 2.2. At most one of xy, xz and yz is an edge in X and Y. So,

$$\mathcal{E}(G) = \mathcal{E}(H) + 1 \leq \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

This completes the proof of the theorem. $\square$

We now characterize the extremal graphs. Through the proof, we notice that the only time we have equality is in case when G obtained by adding an edge to the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. This gives rise to the class $\Omega(G)$.

§3. Edge-Maximal C_5-Disjoint Free Graphs

In this section we determine $f(n; E_{2k+1})$ and characterize the edge maximal members in $\mathcal{G}(n; E_{2k+1})$ for $k = 2$. We now establish that equality (1) holds for $k = 2$. Further we characterize the edge maximal members in $\mathcal{G}(n; E_5)$. To do that we employ the same method as in the above theorem

Theorem 3.1 *Let $G \in \mathcal{G}(n, E_5)$. For $n \geq 9$,*

$$f(n; E_5) \leq \lfloor n^2/4 \rfloor + 1.$$

Furthermore, equality holds if and only if $G \in \Omega(G)$.

Proof Let $G \in \mathcal{G}(n; E_5)$. If G does not have a cycle of length 5, then by Lemma 1.1, $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor$. Thus, $\mathcal{E}(G) < \lfloor n^2/4 \rfloor + 1$. Hence, we consider the case when G has cycles of length 5. Assume $x_1x_2 \ldots x_5x_1$ be a cycle of length 5 in G. As in the proof of the above theorem, we consider $H = G - \{e_1 = x_1x_2, e_2 = x_2x_3, \ldots, e_5 = x_5x_1\}$. Observe that H cannot have 5-cycles as otherwise G would have two 5 - edges disjoint cycles. Now, we consider two cases.

Case 1: H is not a bipartite graph. Then, by Theorem 1.3, we have

$$\mathcal{E}(H) \leq \lfloor (n-2)^2/4 \rfloor + 1.$$

Now,

$$\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}(H) + 5 \\
&\leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + 8 < \left\lfloor \frac{n^2}{4} \right\rfloor - n + 9,
\end{aligned}$$

for $n \geq 9$, we have

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

Case 2: H is a bipartite graph. Let X and Y be the bipartition of $V(H)$. Thus, $\mathcal{E}(G) \leq |X||Y|$. Observe $|X| + |Y| = n$. The maximum of the above is when $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$. Thus, $\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor$. Now, we consider the following subcases.

Subcase 2.1. One of X and Y contains two edges. Observe that those two edges must be consecutive, say e_1 and e_2. Let z be a vertex in X. If $|N_Y(x_1) \cap N_Y(x_2) \cap N_Y(x_3) \cap N_Y(z)| \geq 4$, then G contains two 5 edges disjoint cycles. Thus,

$$\mathcal{E}(\{x_1, x_2, x_3, z\}, Y) \leq 3|Y| + 3.$$

So,

$$\begin{aligned}
\mathcal{E}(G) &= \mathcal{E}(X - x_1, x_2, x_3, z\}, Y) + \mathcal{E}(x_1, x_2, x_3, z\}, Y) + \mathcal{E}(X) + \mathcal{E}(Y) \\
&\leq (|X| - 4)|Y| + 3|Y| + 3 + 2 + 3 \\
&\leq |X||Y| - |Y| + 8 \leq (|X| - 1)|Y| + 8
\end{aligned}$$

Observe $|X| + |Y| = n$. The maximum of the above equation is when $|Y| = \lceil \frac{n-1}{2} \rceil$ and $|X| - 1 = \lfloor \frac{n-1}{2} \rfloor$. Thus,

$$\mathcal{E}(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 8$$

For $n \geq 9$, we have

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

Subcase 2.2. $\mathcal{E}(X) = 1$ and $\mathcal{E}(Y) = 0$ or $\mathcal{E}(X) = 0$ and $\mathcal{E}(Y) = 1$. Then

$$\mathcal{E}(G) \leq \mathcal{E}(H) + 1 \leq \left\lfloor \frac{n^2}{4} \right\rfloor + 1$$

This completes the proof of the theorem. □

We now characterize the extremal graphs. Through the proof, we notice that the only time we have equality is in case when G obtained by adding an edge to the complete bipartite graph. This gives rise to the class $\Omega(G)$.

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A Note on Admissible Mannheim Curves in Galilean Space G_3

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Abstract: In this paper, the definition of the Mannheim partner curves in Galilean space G_3 is given. The relationship between the curvatures and the torsions of the Mannheim partner curves with respect to each other are obtained.

Key Words: Galilean space, Mannheim curve, admissible curve.

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§1. Introduction

The theory of curves is a fundamental structure of differential geometry. An increasing interest of the theory curves makes a development of special curves to be examined. A way for classification and characterization of curves is the relationship between the Frenet vectors of the curves. The well known of such curve is Bertrand curve which is characterized as a kind of corresponding relation between the two curves. A Bertrand curve is defined as a spatial curve which shares its principal normals with another spatial curve (called Bertrand mate). Another kind of associated curve has been called Mannheim curve and Mannheim partner curve. A space curve whose principal normal is the binormal of another curve is called Mannheim curve. The notion of Mannheim curve was discovered by A. Mannheim in 1878. The articles concerning Mannheim curves are rather few. In [1], R. Blum studied a remarkable class of Mannheim curves. O. Tigano obtained general Mannheim curves in Euclidean 3-space in [2]. Recently, H.Liu and F. Wang studied the Mannheim partner curves in Euclidean 3-space and Minkowski 3-space and obtained the necessary and sufficient conditions for the Mannheim partner curves in [3]. On the other hand, a range of new types of geometries have been invented and developed in the last two centuries. They can be introduced in a variety of manners. One possible way is through projective manner, where one can express metric properties through projective relations. For this purpose a fixed conic in infinity is taken and all metric relations with respect to the absolute. This approach is due to A. Cayley and F. Klein who noticed that due to the nature of the absolute, various geometries are possible. Among this geometries, there are also Galilean and pseudo-Galilean geometries. They are very important in physics. Galilean space-time plays the same role in non relativistic physics. The Geometry of the Galilean space

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G_3 has treated in detail in Röschel's habilitiation, [4]. Furthermore, Kamenarovic and Sipus studied about Galilean space, [5]-[6]. The properties of the curves in the Galilean space are studied [7]-[9]. The aim of this paper is to study the Mannheim partner curves in Galilean space G_3 and give some characterizations for this curves.

§2. Preliminaries

The Galilean space G_3 is a Cayley-Klein space equipped with the projective metric of signature (0,0,+,+), as in [10]. The absolute figure of the Galilean Geometry consist of an ordered triple $\{w, f, I\}$, where w is the ideal (absolute) plane, f is the line (absolute line) in w and I is the fixed elliptic involution of points of f, [5]. In the non-homogeneous coordinates the similarity group H_8 has the form

$$\begin{aligned} \overline{x} &= a_{11} + a_{12}x \\ \overline{y} &= a_{21} + a_{22}x + a_{23}y\cos\varphi + a_{23}z\sin\varphi \\ \overline{z} &= a_{31} + a_{32}x - a_{23}y\sin\varphi + a_{23}z\cos\varphi \end{aligned} \tag{2.1}$$

where a_{ij} and φ are real numbers, [7].

In what follows the coefficients a_{12} and a_{23} will play the special role. In particular, for $a_{12} = a_{23} = 1$, (2.1) defines the group $B_6 \subset H_8$ of isometries of Galilean space G_3.

In G_3 there are four classes of lines:

i) (proper) non-isotropic lines- they don't meet the absolute line f.

ii) (proper) isotropic lines- lines that don't belong to the plane w but meet the absolute line f.

iii) un-proper non-isotropic lines-all lines of w but f.

iv) the absolute line f.

Planes x =constant are Euclidean and so is the plane w. Other planes are isotropic, [6].

The Galilean scalar product can be written as

$$< u_1, u_2 >= \begin{cases} x_1x_2 & , \ if \ \ x_1 \neq 0 \ \vee \ x_2 \neq 0 \\ y_1y_2 + z_1z_2 \ , & if \ \ x_1 = 0 \ \wedge \ x_2 = 0 \end{cases}$$

where $u_1 = (x_1, y_1, z_1)$ and $u_2 = (x_2, y_2, z_2)$. It leaves invariant the Galilean norm of the vector $u = (x, y, z)$ defined by

$$\|u\| = \begin{cases} x & , \ x \neq 0 \\ \sqrt{y^2 + z^2} & , \ x = 0. \end{cases}$$

Let α be a curve given in the coordinate form

$$\left(\begin{array}{l} \alpha : I \to G_3, \qquad I \subset \mathbb{R} \\ t \to \alpha(t) = (x(t), y(t), z(t) \end{array} \right) \tag{2.2}$$

where $x(t), y(t), z(t) \in C^3$ and t is a real interval. If $x'(t) \neq 0$, then the curve α is called admissible curve.

Let α be an admissible curve in G_3 parameterized by arc length s and given by

$$\alpha(s) = (s, y(s), z(s))$$

where the curvature $\kappa(s)$ and the torsion $\tau(s)$ are

$$\kappa(s) = \sqrt{y''^2(s) + z''^2(s)} \quad \text{and} \quad \tau(s) = \frac{\det\left[\alpha'(s), \alpha''(s), \alpha'''(s)\right]}{\kappa^2(s)}, \tag{2.3}$$

respectively. The associated moving Frenet frame is

$$\begin{aligned} T(s) &= \alpha'(s) = (1, y'(s), z'(s)) \\ N(s) &= \tfrac{1}{\kappa(s)}\alpha'(s) = \tfrac{1}{\kappa(s)}(0, y''(s), z''(s)) \\ B(s) &= \tfrac{1}{\kappa(s)}(0, -z''(s), y''(s)). \end{aligned} \tag{2.4}$$

Here T, N and B are called the tangent vector, principal normal vector and binormal vector fields of the curve α, respectively. Then for the curve α, the following Frenet equations are given by

$$\begin{aligned} T'(s) &= \kappa(s)N(s) \\ N'(s) &= \tau(s)B(s) \\ B'(s) &= -\tau(s)N(s) \end{aligned} \tag{2.5}$$

where T, N, B are mutually orthogonal vectors, [6].

§3. Admissible Mannheim Curves in Galilean Space G_3

In this section, we defined the admissible Mannheim curve and gave some theorems related to these curves in G_3.

Definition 3.2 *Let α and α^* be an admissible curves with the Frenet frames along $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively. The curvature and torsion of α and α^*, respectively, $\kappa(s), \tau(s)$ and $\kappa^*(s), \tau^*(s)$ never vanish for all $s \in I$ in G_3. If the principal normal vector field N of α coincidence with the binormal vector field B^* of α^* at the corresponding points of the admissible curves α and α^*. Then α is called an admissible Mannheim curve and α^* is an admissible Mannheim mate of α. Thus, for all $s \in I$*

$$\alpha^*(s) = \alpha(s) + \lambda(s)N(s). \tag{3.1}$$

The mate of an admissible Mannheim curve is denoted by (α, α^*), (see Figure 1).

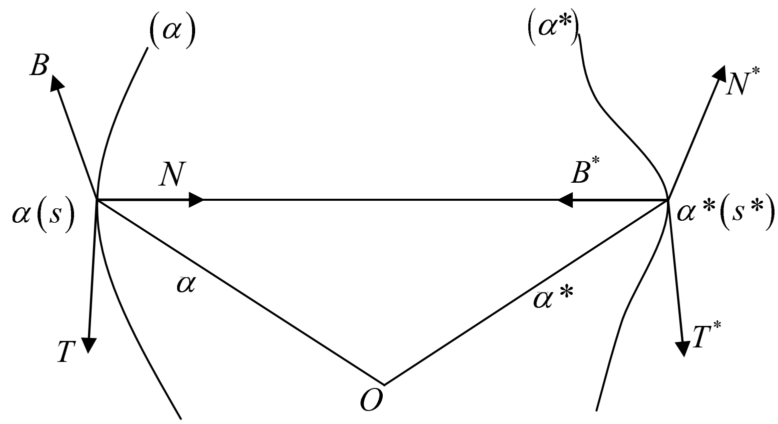

Figure 1. The admissible Mannheim partner curves

Theorem 3.1 *Let (α, α^*) be a mate of admissible Mannheim pair in G_3. Then function λ is constant.*

Proof Let α be an admissible Mannheim curve in G_3 and α^* be an admissible Mannheim mate of α. Let the pair of $\alpha(s)$ and $\alpha^*(s)$ be corresponding points of α and α^*. Then the curve $\alpha^*(s)$ is given by (3.1). Differentiating (3.1) with respect to s and using Frenet equations,

$$T^* \frac{ds^*}{ds} = T + \lambda' N + \lambda\tau B \tag{3.2}$$

is obtained.

Here and here after prime denotes the derivative with respect to s. Since N is coincident with B^* in the same direction, we have

$$\lambda'(s) = 0, \tag{3.3}$$

that is, λ is constant. This theorem proves that the distance between the curve α and its Mannheim mate α^* is constant at the corresponding points of them. It is notable that if α^* is an admissible Mannheim mate of α. Then α is also Mannheim mate of α^* because the relationship obtained in theorem between a curve and its Mannheim mate is reciprocal one. □

Theorem 3.2 *Let α be an admissible curve with arc length parameter s. The curve α is an admissible Mannheim curve if and only if the torsion τ of α is constant.*

Proof Let (α, α^*) be a mate of an admissible Mannheim curves, then there exists the relation

$$\begin{aligned} T(s) &= \cos\theta T^*(s) + \sin\theta N^*(s) \\ B(s) &= -\sin\theta T^*(s) + \cos\theta N^*(s) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} T^*(s) &= \cos\theta T(s) - \sin\theta B(s) \\ N^*(s) &= \sin\theta T(s) + \cos\theta B(s) \end{aligned} \tag{3.5}$$

where θ is the angle between T and T^* at the corresponding points of $\alpha(s)$ and $\alpha^*(s)$, (see Figure 1).

By differentiating (3.5) with respect to s, we get

$$\tau^* B^* \tfrac{ds^*}{ds} = \tfrac{d(\sin\theta)}{ds} T + \sin\theta\kappa N + \cos\theta\tau N + \tfrac{d(\cos\theta)}{ds} B. \tag{3.6}$$

Since the principal normal vector field N of the curve α and the binormal vector field B^* of its Mannheim mate curve, then it can be seen that θ is a constant angle.

If the equations (3.2) and (3.5) is considered, then

$$\lambda\tau\cot\theta = 1 \tag{3.7}$$

is obtained. According to Theorem 3.1 and constant angle θ, $u = \lambda\cot\theta$ is constant. Then from equation (3.7), $\tau = \frac{1}{u}$ is constant, too. Hence the proof is completed. □

Theorem 3.3(Schell's Theorem) *Let (α, α^*) be a mate of an admissible Mannheim curves with torsions τ and τ^*, respectively. The product of torsions τ and τ^* is constant at the corresponding points $\alpha(s)$ and $\alpha^*(s)$.*

Proof Since α is an admissible Mannheim mate of α^*, then equation (3.1) also can be given by

$$\alpha = \alpha^* - \lambda B^*. \tag{3.8}$$

By taking differentiation of last equation and using equation (3.4),

$$\tau^* = \tfrac{1}{\lambda}\tan\theta \tag{3.9}$$

can be given. By the helps of (3.7), the equation below is easily obtained;

$$\tau\tau^* = \tfrac{\tan^2\theta}{\lambda^2} = constant \tag{3.10}$$

This completes the proof. □

Theorem 3.4 *Let (α, α^*) be an admissible Mannheim mate with curvatures κ, κ^* and torsions τ, τ^* of α and α^*, respectively. Then their curvatures and torsions satisfy the following relations*

$$\begin{aligned} &i)\ \ \kappa^* = -\tfrac{d\theta}{ds^*} \\ &ii)\ \ \kappa = \tau^* \tfrac{ds^*}{ds}\sin\theta \\ &iii)\ \tau = -\tau^* \tfrac{ds^*}{ds}\cos\theta. \end{aligned}$$

Proof i) Let us consideration equation (3.4), then we have

$$< T, T^* >= \cos\theta. \tag{3.11}$$

By differentiating last equation with respect to s^* and using the Frenet equations of α and α^*, we reach

$$< \kappa(s)N(s)\tfrac{ds}{ds^*}, T^*(s) > + < T(s), \kappa^*(s)N^*(s) >= -\sin\theta\tfrac{d\theta}{ds^*}. \tag{3.12}$$

Since the principal normal N of α and binormal B^* of α^* are linearly dependent. By considering equations (3.4) and (3.12), we reach

$$\kappa^*(s) = -\tfrac{d\theta}{ds^*}. \tag{3.13}$$

If we take into consideration $< T, B^* >$, $< B, B^* >$ scalar products and (2.5), (3.4), (3.5) equations, then we can easily prove *ii)* and *iii)* items of the theorem, respectively.
The relations given in *ii)* and *iii)* of the last theorem, we obtain

$$\frac{\kappa}{\tau} = -\tan\theta = constant.$$

□

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The Number of Spanning Trees in Generalized Complete Multipartite Graphs of Fan-Type

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Abstract: Let $K_{k,\overline{n}}$ be a complete k-partite graph of order n and let $K^{\mathscr{F}}_{k,\overline{n}}$ be a generalized complete k-partite graph of order n spanned by the fan set $\mathscr{F} = \{F_{n_1}, F_{n_2}, \cdots, F_{n_k}\}$, where $\overline{n} = \{n_1, n_2, \cdots, n_k\}$ and $n = n_1 + n_2 + \cdots + n_k$ for $1 \leqslant k \leqslant n$. In this paper, we get the number of spanning trees in $K_{k,\overline{n}}$ to be

$$t(K_{k,\overline{n}}) = n^{k-2}\prod_{i=1}^{k}(n-n_i)^{n_i-1}.$$

and the number of spanning trees in $K^{\mathscr{F}}_{k,\overline{n}}$ to be

$$t(K^{\mathscr{F}}_{k,\overline{n}}) = n^{2k-2}\prod_{i=1}^{k}\frac{\alpha_i^{n_i-1} - \beta_i^{n_i-1}}{\alpha_i - \beta_i}$$

where $\alpha_i = (d_i + \sqrt{d_i^2 - 4})/2$ and $\beta_i = (d_i - \sqrt{d_i^2 - 4})/2, d_i = n - n_i + 3$. In particular, $K_{1,\overline{n}} = K_n^c$ with $t(K_{1,\overline{n}}) = 0$, $K_{n,\overline{n}} = K_n$ with $t(K_{n,\overline{n}}) = n^{n-2}$ which is just the Cayley's formula and $K^{\mathscr{F}}_{1,\overline{n}} = F_n$ with $t(K^{\mathscr{F}}_{1,\overline{n}}) = (\alpha^{n-1} - \beta^{n-1})/\sqrt{5}$ where $\alpha = (3+\sqrt{5})/2$ and $\beta = (3-\sqrt{5})/2$ which is just the formula given by Z.R.Bogdanowicz in 2008.

Key Words: Connected simple graph, k-partite graph, complete graph, tree, Smarandache (E_1, E_2)-number of trees.

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§1. Introduction

Graphs considered here are simple finite and undirected. A graph is *simple* if it contains neither multiple edges nor loops. A graph is denoted by $G = \langle V(G), E(G)\rangle$ with n vertices and m edges where $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $E(G) = \{e_1, e_2, \cdots, e_m\}$ denote the sets of its vertices and

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edges, respectively. The *degree* of a vertex v in a graph G is the number of edges incident with v and is denoted by $d(v) = d_G(v)$.

For simple graphs $G_i = \langle V_i(G), E_i(G)\rangle$ with vertex set $V_i = \{v_{i1}, v_{i2}, \cdots, v_{in_i}\}$, the empty graphs of order n_i are denoted by $N_{n_i} = \langle V_i, \phi\rangle$, $i = 1, 2, \cdots, k$. A complete k-partite graph $K_{k,\overline{n}} = K_{n_1,n_2,\cdots,n_k} = \langle V_1, V_2, \cdots, V_k, E\rangle$ is said to be one spanned by the empty graph set $\mathscr{N} = \{N_{n_1}, N_{n_2}, \cdots, N_{n_k}\}$, denoted by $K^{\mathscr{N}}_{n_1,n_2,\cdots,n_k}$or $K^{\mathscr{N}}_{k,\overline{n}}$ where $\overline{n} = \{n_1, n_2, \cdots, n_k\}$.

In generally, for the graph set $\mathscr{G} = \{G_{n_1}, G_{n_2}, \cdots, G_{n_k}\}$, the graph

$$K^{\mathscr{G}}_{k,\overline{n}} = K^{\mathscr{N}}_{k,\overline{n}} \cup G_{n_1} \cup G_{n_2} \cup \cdots \cup G_{n_k} \tag{1}$$

is said to be *a generalized complete k-partite graph* spanned by the graph set $\mathscr{G}$.

For all graph theoretic terminology not described here we refer to [1]. Let G be a connected graph and $E_1, E_2 \subset E(E)$ with $E_1 \neq E_2$. The *Smarandache (E_1, E_2)-number* $t^S(E_1, E_2)$ of trees is the number of such spanning trees T in G with $E(T)\cap E_1 \neq \emptyset$ but $E(T)\cap E_2 = \emptyset$. Particularly, if $E_1 = E(G)$ and $E_2 = \emptyset$, i.e., such number is just the number of labeled spanning trees of a graph G, denoted by $t(G)$. For a few special families of graphs there exist simple formulas that make it much easier to calculate and determine the number of corresponding spanning trees. One of the first such results is due to Cayley [2] who showed in 1889 that complete graph on n vertices, K_n, has n^{n-2} spanning trees. That is

$$t(K_n) = n^{n-2} \quad \text{for} \quad n \geqslant 2. \tag{2}$$

Another result is due to Sedlacek [3] who derived in 1970 a formula for the wheel on $n+1$ vertices, W_{n+1}, which is formed from a cycle C_n on n vertices by adding a new vertex adjacent to every vertex of C_n. That is

$$t(W_{n+1}) = (\frac{3+\sqrt{5}}{2})^n + (\frac{3-\sqrt{5}}{2})^n - 2 \quad \text{for} \quad n \geqslant 3. \tag{3}$$

Sedlacek [4] also later derived a formula for the number of spanning trees in a Möbius ladder, M_n, is formed from a cycle C_{2n} on $2n$ vertices ladder $v_1, v_2, \cdots, v_{2n}$ by adding edges v_iv_{n+i} for every vertex v_i, where $i \leqslant n$. That is

$$t(M_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n + 2] \quad \text{for} \quad n \geqslant 2. \tag{4}$$

In 1985, Baron et al [5] derived the formula for the number of spanning trees in a square of cycle, C^2_n, which is expressed as follows.

$$t(C^2_n) = nF_-\{n\}, \quad n \geqslant 5, \tag{5}$$

where $F_-\{n\}$ is the n'th Fibonacci Number. Similar results can also be found in [6].

The next result is due to Boesch and Bogdanowicz [7] who derived in 1987 a formula for the prism on $2n$ vertices, R_n, which is formed from two disjoint cycles C_n with vertex set $V(C_n) = \{v_1, v_2, \cdots, v_n\}$ and C'_n with vertex set $V(C'_n) = \{v'_1, v'_2, \cdots, v'_n\}$ by adding all edges of the form $v_iv'_i$. That is

$$t(R_n) = \frac{n}{2}[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2] \quad \text{for} \quad n \geqslant 3. \tag{6}$$

In 2007, Bleller and Saccoman [8] derived a formula for a threshold graph on n vertices, T_n, which is formed from that for all pairs of vertices u and v in T_n, $N(u)-v \subset N(v)-u$ whenever $d(u) \leqslant d(v)$. That is

$$t(T_n)=\prod_{i=1}^{k} d_i^{n_i} \prod_{j=k+2}^{s-1}(d_j+1)^{n_j}(d_{k+1}+l)^{n_{k+1}-1}(d_s+1)^{n_s-1}, \tag{7}$$

where $\overline{d}(T_n)=(d_1^{(n_1)}, d_2^{(n_2)}, \cdots, d_s^{(n_s)})$ is the degree sequence of T_n, $d_i<d_{i+1}$ for $i=1,2,\cdots,s-1$, $k=\lfloor\frac{s-1}{2}\rfloor$ and $s \equiv l (\bmod\ 2)$.

In 2008, Bogdanowicz [9] also derived a formula for an n-fan on $n+1$ vertices, F_{n+1}, which is formed from a n-path P_n by adding an additional vertex adjacent to every vertex of P_n. That is

$$t(F_{n+1})=\frac{2}{5-3\sqrt{5}}[(\frac{3-\sqrt{5}}{2})^{n+1}-(\frac{3+\sqrt{5}}{2})^{n-1}] \quad \text{for} \quad n \geqslant 2. \tag{8}$$

In this paper it is proved that: Let $K_{k,\overline{n}}$ be a complete k-partite graph of order n where the vector $\overline{n}=\{n_1,n_2,\cdots,n_k\}$ and $n=n_1+n_2+\cdots+n_k$ for $1 \leqslant k \leqslant n$, then the number of spanning trees in $K_{k,\overline{n}}$ is

$$t(K_{k,\overline{n}})=n^{k-2}\prod_{i=1}^{k}(n-n_i)^{n_i-1}. \tag{9}$$

Moreover, let $K_{k,\overline{n}}^{\mathscr{F}}$ be a generalized complete k-partite graph of order n spanned by the fan set $\mathscr{F}=\{F_{n_1},F_{n_2},\cdots,F_{n_k}\}$ where $\overline{n}=\{n_1,n_2,\cdots,n_k\}$ and $n=n_1+n_2+\cdots+n_k$ for $1 \leqslant k \leqslant n$, then the number of spanning trees in $K_{k,\overline{n}}^{\mathscr{F}}$ is

$$t(K_{k,\overline{n}}^{\mathscr{F}})=n^{2k-2}\prod_{i=1}^{k}\frac{\alpha_i^{n_i-1}-\beta_i^{n_i-1}}{\alpha_i-\beta_i} \tag{10}$$

where $\alpha_i=(d_i+\sqrt{d_i^2-4})/2$ and $\beta_i=(d_i-\sqrt{d_i^2-4})/2$,$d_i=n-n_i+3$.

In particular, from (9) we can obtain $K_{1,\overline{n}}=K_n^c$ with $t(K_n^c)=0$ and $K_{n,\overline{n}}=K_n$ with $t(K_n)=n^{n-2}$ which is just the Cayley's formula (2). From (10) we can also obtain $K_{1,\overline{n}}^{\mathscr{F}}=F_n$ with $t(K_{1,\overline{n}}^{\mathscr{F}})=(\alpha^{n-1}-\beta^{n-1})/\sqrt{5}$ where $\alpha=\frac{1}{2}(3+\sqrt{5})$ and $\beta=\frac{1}{2}(3-\sqrt{5})$ which is the formula (8), too.

§2. Some Lemmas

In order to calculate the number of spanning trees of G, we first denote by $A(G)$, or $A=(a_{ij})_{n\times n}$, the *adjacency matrix* of G, which has the rows and columns corresponding to the vertices, and entries $a_{ij}=1$ if there is an edge between vertices v_i and v_j in $V(G)$, $a_{ij}=0$ otherwise.

In addition, let $D(G)$ represent the diagonal matrix of the degrees of the vertices of G. We denote by $H(G)$, or $H=(h_{ij})_{n\times n}$, the *Laplacian matrix* (also known as the nodal admittance matrix) $D(G)-A(G)$ of G. From $H(G)=D(G)-A(G)$, we can see that $h_{ii}=d(v_i)$ and $h_{ij}=-a_{ij}$ if $i \neq j$.

A well-known result states the relationship between the number of spanning trees of a graph and the eigenvalues of its nodal admittance matrix:

Lemma 2.1([10]) *The value of $t(G)$, the number of spanning trees of a graph G, is related to the eigenvalues $\lambda_1(G), \lambda_2(G), \cdots, \lambda_n(G)$ of its nodal admittance matrix $H = H(G)$ as follows:*

$$t(G) = \frac{1}{n}\prod_{i=2}^{n}\lambda_i(G), \qquad 0 = \lambda_1(G) \leqslant \lambda_2(G) \leqslant \cdots \leqslant \lambda_n(G). \tag{11}$$

A classic result of Kirchhoff can be used to determine the number of spanning trees for G. Next, we state the well-known theorem of Kirchhoff:

Lemma 2.2(Kirchhoff's Matrix-Tree Theorem, [11]) *All cofactors of H are equal and their common value is the number of spanning trees.*

Lemma 2.3 *Let $b > 2$ be a constant, then the determinant of order m*

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & b & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & b & -1 \\ 0 & \cdots & 0 & -1 & b-1 \end{vmatrix}_{m\times m} = \frac{\alpha^m - \beta^m}{\alpha - \beta}, \tag{12}$$

where $\alpha = (b + \sqrt{b^2 - 4})/2$, $\beta = (b - \sqrt{b^2 - 4})/2$.

Proof Let a_m stand for the determinant in (12) as above, by expending the determinant according to the first column we then have

$$a_m = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & b & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & b & -1 \\ 0 & \cdots & 0 & -1 & b-1 \end{vmatrix}_{m\times m} = c_{m-1} + a_{m-1}, \tag{13}$$

where

$$c_m = \begin{vmatrix} b & -1 & 0 & \cdots & 0 \\ -1 & b & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & b & -1 \\ 0 & \cdots & 0 & -1 & b-1 \end{vmatrix}_{m\times m} = bc_{m-1} - c_{m-2} \tag{14}$$

in which $c_0 = 1$, $c_1 = b - 1$ and $c_2 = b^2 - b - 1$.

Constructing a function as follows:

$$f(x)=\sum_{m\geqslant 0} c_m x^m, \tag{15}$$

then

$$\begin{aligned} f(x) &= 1+(b-1)x+\sum_{m\geqslant 2} c_m x^m \\ &= 1+(b-1)x+\sum_{m\geqslant 2}(bc_{m-1}-c_{m-2})x^m \\ &= 1+(b-1)x+bx(f(x)-1)-x^2f(x), \end{aligned}$$

i.e.

$$f(x)=\frac{1-x}{x^2-bx+1}=\sum_{m\geqslant 0}\frac{\alpha^{m+1}+\beta^m}{1+\alpha}x^m \tag{16}$$

where $\alpha=(b+\sqrt{b^2-4})/2$ and $\beta=(b-\sqrt{b^2-4})/2$ which is just the two solutions of the equation $x^2-bx+1=0$.

Thus, from (11.4) and (11.5) we have

$$c_m=\frac{\alpha^{m+1}+\beta^m}{1+\alpha}. \tag{17}$$

Since

$$a_3=\begin{vmatrix} 1 & 1 & 1 \\ -1 & b & -1 \\ 0 & -1 & b-1 \end{vmatrix}=b^2-1=c_2+c_1+c_0, \tag{18}$$

from (11.2),(11.6) and (11.7) we can obtain that

$$a_m=\sum_{k=0}^{m-1}c_k=\sum_{k=0}^{m-1}\frac{\alpha^{k+1}+\beta^k}{1+\alpha}=\frac{\alpha^m-\beta^m}{\alpha-\beta}.$$

§3 Complete Multipartite Graphs

For a complete k-partite graph $K_{k,\overline{n}}$ of order n where the vector $\overline{n}=\{n_1,n_2,\cdots,\ n_k\}$ and $n=n_1+n_2+\cdots+n_k$ for $1\leqslant k\leqslant n$ with $n_1\geqslant n_2\geqslant\cdots\geqslant n_k$. Let $V(K_{k,\overline{n}})=V_1\cup V_2\cup\cdots\cup V_k$ be the k partitions of the graph $K_{k,\overline{n}}$ such that $|V_i|=n_i$ for $i=1,2,\cdots,k$. It is clear that for any vertex $v_i\in V_i$, $d(v_i)=n-n_i$ for $i=1,2,\cdots,k$. So the degree sequence of vertices in $K_{k,\overline{n}}$

$$\overline{d}(K_{k,\overline{n}})=((n-n_1)^{n_1},(n-n_2)^{n_2},\cdots,(n-n_k)^{n_k}). \tag{19}$$

Now, let E_i be an identity matrix of order i, $I_{i\times j}=(1)_{i\times j}$ a total module matrix of order $i\times j$ and $O_{i\times j}=(0)_{i\times j}$ a total zero matrix of order $i\times j$. Then the diagonal block matrix of the degrees of the vertices of $K_{k,\overline{n}}$ corresponding to (12) is

$$D(K_{k,\overline{n}})=(D_{ij})_{k\times k}, \tag{20}$$

where $D_{ii} = (n - n_i)E_{n_i}$ and $D_{ij} = O_{n_i \times n_j}$ when $i \neq j$ for $1 \leqslant i, j \leqslant k$. The adjacency matrix of $K_{k,\overline{n}}$ corresponding to (19) is

$$A(K_{k,\overline{n}}) = (A_{ij})_{k \times k}, \tag{21}$$

where $A_{ii} = O_{n_i}$ and $A_{ij} = I_{n_i \times n_j}$ when $i \neq j$ for $1 \leqslant i, j \leqslant k$.

Thus, we have from (13) and (14) the Laplacian block matrix (or the nodal admittance matrix) of $K_{k,\overline{n}}$

$$H(K_{k,\overline{n}}) = D(K_{k,\overline{n}}) - A(K_{k,\overline{n}}) = (H_{ij})_{k \times k}, \tag{22}$$

where $H_{ii} = (n - n_i)E_{n_i}$ and $H_{ij} = -I_{n_i \times n_j}$ when $i \neq j$ for $1 \leqslant i, j \leqslant k$. □

Theorem 3.1 *Let $K_{k,\overline{n}}$ be a complete k-partite graph of order n where the vector $\overline{n} = \{n_1, n_2, \cdots, n_k\}$ and $n = n_1 + n_2 + \cdots + n_k$ for $1 \leqslant k \leqslant n$ with $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$. Then the number of spanning trees of $K_{k,\overline{n}}$ is*

$$t(K_{k,\overline{n}}) = n^{k-2} \prod_{i=1}^{k} (n - n_i)^{n_i - 1}. \tag{23}$$

Proof According to Lemma 2.1, we need to determine all eigenvalues of nodal admittance matrix $H(K_{k,\overline{n}})$ of $K_{k,\overline{n}}$. From (15) we can get easily the characteristic polynomial of $H(K_{k,\overline{n}})$ as follows:

$$|H(K_{k,\overline{n}}) - \lambda E| =$$

$$\begin{vmatrix} (n - n_1 - \lambda)E_{n_1} & -I_{n_1 \times n_2} & \cdots & -I_{n_1 \times n_k} \\ -I_{n_2 \times n_1} & (n - n_2 - \lambda)E_{n_2} & \cdots & -I_{n_2 \times n_k} \\ \cdots & \cdots & \cdots & \cdots \\ -I_{n_k \times n_1} & -I_{n_k \times n_2} & \cdots & (n - n_k - \lambda)E_{n_k} \end{vmatrix}_{n \times n}. \tag{24}$$

Since the summations of entries in every column in $|H(K_{k,\overline{n}}) - \lambda E|$ all are $-\lambda$, by adding the entries of all rows other than the first row to the first row in $|H(K_{k,\overline{n}}) - \lambda E|$, all entries of the first row then become $-\lambda$. Thus, the determinant becomes

$$|H(K_{k,\overline{n}}) - \lambda E| =$$

$$-\lambda \begin{vmatrix} H^+_{n_1} & * & \cdots & * \\ -I_{n_2 \times n_1} & (n - n_2 - \lambda)E_{n_2} & \cdots & -I_{n_2 \times n_k} \\ \cdots & \cdots & \cdots & \cdots \\ -I_{n_k \times n_1} & -I_{n_k \times n_2} & \cdots & (n - n_k - \lambda)E_{n_k} \end{vmatrix}_{n \times n}, \tag{25}$$

where

$$H^+_{n_1} = \begin{pmatrix} 1 & I_{1 \times (n_1 - 1)} \\ O_{(n_1 - 1) \times 1} & (n - n_1 - \lambda)E_{(n_1 - 1)} \end{pmatrix}_{n_1 \times n_1} \tag{26}$$

and the stars "*" in (18) stand for the matrices with all entries in the first row being 1. By adding the entries in the first row to the rows from row n_1+1 to row n in the determinant (18), it then becomes

$$|H(K_{k,\overline{n}})-\lambda E| =$$

$$-\lambda \begin{vmatrix} H^+_{n_1} & * & \cdots & * \\ O_{n_2\times n_1} & (n-n_2-\lambda)E_{n_2}+I_{n_2} & \cdots & O_{n_2\times n_k} \\ \cdots & \cdots & \cdots & \cdots \\ O_{n_k\times n_1} & O_{n_k\times n_2} & \cdots & (n-n_k-\lambda)E_{n_k}+I_{n_k} \end{vmatrix}_{n\times n},$$

i.e.

$$|H(K_{k,\overline{n}})-\lambda E| = -\lambda |H^+_{n_1}| \prod_{1<i\leqslant k} |(n-n_i-\lambda)E_{n_i}+I_{n_i}|. \quad (27)$$

From (19) we have

$$|H^+_{n_1}| = \begin{vmatrix} 1 & I_{1\times n_1} \\ O_{n_1\times 1} & (n-n_1-\lambda)E_{n_1} \end{vmatrix} = (n-n_1-\lambda)^{n_1-1}. \quad (28)$$

For $1<i\leqslant k$ we have

$$|(n-n_i-\lambda)E_{n_i}+I_{n_i}| = \begin{vmatrix} n-\lambda & (n-\lambda)I_{1\times(n_i-1)} \\ I_{(n_i-1)\times 1} & (n-n_i-\lambda)E_{n_i-1}+I_{n_i-1} \end{vmatrix}$$

i.e.

$$\begin{aligned} |(n-n_i-\lambda)E_{n_i}+I_{n_i}| &= (n-\lambda)\begin{vmatrix} 1 & I_{1\times(n_i-1)} \\ O_{(n_i-1)\times 1} & (n-n_i-\lambda)E_{n_i-1} \end{vmatrix} \\ &= (n-\lambda)(n-n_i-\lambda)^{n_i-1}. \end{aligned} \quad (29)$$

By substituting (21),(22) to (20) we have

$$|H(K_{k,\overline{n}})-\lambda E| = -\lambda(n-\lambda)^{k-1}\prod_{i=1}^{k}(n-n_i-\lambda)^{n_i-1}. \quad (30)$$

So, from (23) we derive all eigenvalues of $H(K_{k,\overline{n}})$ as follows: $\lambda_1=0$ and

$$\begin{cases} (n_i-1)\text{-multiple roots } \ \lambda_{i+1}(K_{k,\overline{n}}) = n-n_i, \quad i=1,2,\cdots,k; \\ \\ (k-1)\text{-multiple root } \ \lambda_n(K_{k,\overline{n}}) = n. \end{cases} \quad (31)$$

By substituting (24) to (11) we have

$$t(K_{k,\overline{n}}) = \frac{1}{n}\prod_{i=2}^{n}\lambda_i(K_{k,\overline{n}}) = n^{k-2}\prod_{i=1}^{k}(n-n_i)^{n_i-1}. \quad (32)$$

This is just the theorem. □

Corollary 3.2 (Cayley's formula) *The total number of spanning trees of complete graph K_n is*

$$t(K_n) = n^{n-2}. \tag{33}$$

Proof Let $k = n$, then $n_1 = n_2 = \cdots = n_n = 1$ and $K_{n,\overline{n}} = K_n$. By substituting it to (25) we then have

$$t(K_n) = t(K_{n,\overline{n}}) = n^{n-2}\prod_{i=1}^{n}(n-1)^0 = n^{n-2}.$$ □

§3. Generalized Complete Multipartite Graphs

For a generalized complete k-partite graph $K^{\mathscr{F}}_{k,\overline{n}}$ of order n spanned by the fan set $\mathscr{F} = \{F_{n_1}, F_{n_2}, \cdots, F_{n_k}\}$ where $n = n_1 + n_2 + \cdots + n_k$ with $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$ for $1 \leqslant k \leqslant n$. It is clear that the degree sequence of vertices of the fan F_{n_i} in $K^{\mathscr{F}}_{k,\overline{n}}$

$$\overline{d}(F_{n_i}) = (n - n_i + 2, (n - n_i + 3)^{n_i - 3}, n - n_i + 2, n - 1) \tag{34}$$

for $i = 1, 2, \cdots, k$. So the degree sequence of vertices of $K^{\mathscr{F}}_{k,\overline{n}}$ is

$$\overline{d}(K^{\mathscr{F}}_{k,\overline{n}}) = (\overline{d}(F_{n_1}), \overline{d}(F_{n_2}), \cdots, \overline{d}(F_{n_k})). \tag{35}$$

Now, the diagonal block matrix of the degrees of the vertices of $K^{\mathscr{F}}_{k,\overline{n}}$ corresponding to (28) is

$$D(K^{\mathscr{F}}_{k,\overline{n}}) = (D_{ij})_{k\times k}, \tag{36}$$

where from (27) we have

$$D_{ii} = \mathrm{diag}\{n - n_i + 2, n - n_i + 3, \cdots, n - n_i + 3, n - n_i + 2, n - 1\} \tag{37}$$

and $D_{ij} = O_{n_i \times n_j}$ when $i \neq j$ for $1 \leqslant i, j \leqslant k$.

The adjacency matrix of $K^{\mathscr{F}}_{k,\overline{n}}$ corresponding to (28) is

$$A(K^{\mathscr{F}}_{k,\overline{n}}) = (A_{ij})_{k\times k}, \tag{38}$$

where

$$A_{ii} = \begin{pmatrix} 0 & 1 & & & 1 \\ 1 & 0 & \ddots & & 1 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}_{n_i \times n_i} \tag{39}$$

and $A_{ij} = I_{n_i \times n_j}$ when $i \neq j$ for $1 \leqslant i, j \leqslant k$.

Thus, we have from (29) $\sim$ (32) the Laplacian block matrix (or the nodal admittance matrix) of $K_{k,\overline{n}}^{\mathscr{F}}$

$$H(K_{k,\overline{n}}^{\mathscr{F}}) = D(K_{k,\overline{n}}^{\mathscr{F}}) - A(K_{k,\overline{n}}^{\mathscr{F}}) = (H_{ij})_{k\times k}, \tag{40}$$

where

$$H_{ii} = D_{ii} - A_{ii} =$$

$$\begin{pmatrix} d_i-1 & -1 & & & & -1 \\ -1 & d_i & -1 & & & -1 \\ & \ddots & \ddots & \ddots & & \vdots \\ & & -1 & d_i & -1 & \\ & & & -1 & d_i-1 & -1 \\ -1 & -1 & \cdots & & -1 & n-1 \end{pmatrix}_{n_i\times n_i} \tag{41}$$

with $d_i = n - n_i + 3$ and $H_{ij} = -I_{n_i\times n_j}$ when $i \neq j$ for $1 \leqslant i, j \leqslant k$.

Theorem 4.1 *Let $K_{k,\overline{n}}^{\mathscr{F}}$ be a generalized complete k-partite graph of order n spanned by the fan set $\mathscr{F} = \{F_{n_1}, F_{n_2}, \cdots, F_{n_k}\}$ where $\overline{n} = \{n_1, n_2, \cdots, n_k\}$ and $n = n_1 + n_2 + \cdots + n_k$ for $1 \leqslant k \leqslant n$, then the number of spanning trees in $K_{k,\overline{n}}^{\mathscr{F}}$ is*

$$t(K_{k,\overline{n}}^{\mathscr{F}}) = n^{2k-2}\prod_{i=1}^{k}\frac{\alpha_i^{n_i-1} - \beta_i^{n_i-1}}{\alpha_i - \beta_i} \tag{42}$$

where $\alpha_i = (d_i + \sqrt{d_i^2 - 4})/2$ and $\beta_i = (d_i - \sqrt{d_i^2 - 4})/2$,$d_i = n - n_i + 3$.

Proof According to the Kirchhoff's Matrix-Tree theorem, all cofactors of $H(K_{k,\overline{n}}^{\mathscr{F}})$ are equal to the number of spanning trees $t(K_{k,\overline{n}}^{\mathscr{F}})$. Let $H^*(K_{k,\overline{n}}^{\mathscr{F}})$ be the cofactor of $H(K_{k,\overline{n}}^{\mathscr{F}})$ corresponding to the entry h_{nn} of $H(K_{k,\overline{n}}^{\mathscr{F}})$, then

$$t(K_{k,\overline{n}}^{\mathscr{F}}) = |H^*(K_{k,\overline{n}}^{\mathscr{F}})| =$$

$$\begin{vmatrix} H_{11} & \cdots & -I_{n_1\times n_i} & \cdots & -I_{n_1\times(n_k-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -I_{n_i\times n_1} & \cdots & H_{ii} & \cdots & -I_{n_i\times(n_k-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -I_{(n_k-1)\times n_1} & \cdots & -I_{(n_k-1)\times n_i} & \cdots & H_{kk}^* \end{vmatrix} \tag{43}$$

where

$$H^*_{kk}=\begin{pmatrix} d_k-1 & -1 & & & \\ -1 & d_k & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & d_k & -1 \\ & & & -1 & d_k-1 \end{pmatrix}_{(n_k-1)\times(n_k-1)}. \tag{44}$$

Since the summations of entries in every column in $|H^*(K^{\mathscr{F}}_{k,\overline{n}})|$ all are 1 by adding the entries of all rows other than the first row to the first row in $|H^*(K^{\mathscr{F}}_{k,\overline{n}})|$ the entries of the first row become 1. By adding the entries in the first row to the rows from row n_1+1 to row $n-1$ in the last determinant, it becomes from (36) and (37)

$$|H^*(K^{\mathscr{F}}_{k,\overline{n}})|=\begin{vmatrix} H^*_{n_1} & \cdots & * & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ O_{n_i\times n_1} & \cdots & H_{ii}+I_{n_i} & \cdots & * \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ O_{(n_k-1)\times n_1} & \cdots & O_{(n_k-1)\times n_i} & \cdots & H^*_{n_k-1} \end{vmatrix},$$

i.e.

$$t(K^{\mathscr{F}}_{k,\overline{n}})=|H^*(K^{\mathscr{F}}_{k,\overline{n}})|=|H^*_{n_1}|\cdot\prod_{1<i<k}|H_{ii}+I_{n_i}|\cdot|H^*_{n_k-1}|, \tag{45}$$

where

$$|H^*_{n_1}|=\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & d_1 & -1 & & & -1 \\ & \ddots & \ddots & \ddots & & \vdots \\ & & -1 & d_1 & -1 & -1 \\ & & & -1 & d_1-1 & -1 \\ -1 & -1 & \cdots & -1 & -1 & n-1 \end{vmatrix}_{n_1\times n_1},$$

by adding the entries in the first row to the correspondence entries of the other row in $|H^*_{n_1}|$ we then have

$$|H^*_{n_1}|=\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & d_1+1 & 0 & 1 & \vdots & 0 \\ 1 & \ddots & \ddots & \ddots & 1 & \vdots \\ \vdots & 1 & 0 & d_1+1 & 0 & 0 \\ 1 & \cdots & 1 & 0 & d_1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & n \end{vmatrix}_{n_1\times n_1},$$

i.e.

$$|H^*_{n_1}| = n \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & d_1+1 & 0 & 1 & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & 0 & d_1+1 & 0 \\ 1 & \cdots & 1 & 0 & d_1 \end{vmatrix}_{(n_1-1)\times(n_1-1)} ; \tag{46}$$

It is clear that

$$|H_{ii} + I_{n_i}| = \begin{vmatrix} d_i & 0 & 1 & \cdots & 1 & 0 \\ 0 & d_i+1 & 0 & 1 & \vdots & \\ 1 & \ddots & \ddots & \ddots & 1 & \vdots \\ \vdots & 1 & 0 & d_i+1 & 0 & \\ 1 & \cdots & 1 & 0 & d_i & 0 \\ 0 & & \cdots & & 0 & n \end{vmatrix}_{n_i\times n_i} ,$$

i.e.

$$|H_{ii} + I_{n_i}| = n \begin{vmatrix} d_i & 0 & 1 & \cdots & 1 \\ 0 & d_i+1 & 0 & 1 & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & 0 & d_i+1 & 0 \\ 1 & \cdots & 1 & 0 & d_i \end{vmatrix}_{(n_i-1)\times(n_i-1)} .$$

Since the summations of all entries in every column in the last determinant, we have

$$|H_{ii} + I_{n_i}| = n^2 \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & d_i+1 & 0 & 1 & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & 0 & d_i+1 & 0 \\ 1 & \cdots & 1 & 0 & d_i \end{vmatrix}_{(n_i-1)\times(n_i-1)} ; \tag{47}$$

and

$$|H^*_{n_k-1}| = \begin{vmatrix} d_k & 0 & 1 & \cdots & 1 \\ 0 & d_k+1 & 0 & 1 & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & 0 & d_k+1 & 0 \\ 1 & \cdots & 1 & 0 & d_k \end{vmatrix}_{(n_k-1)\times(n_k-1)} ,$$

Similarly, we have

$$|H^*_{n_k-1}| = n \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & d_k+1 & 0 & 1 & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & 0 & d_k+1 & 0 \\ 1 & \cdots & 1 & 0 & d_k \end{vmatrix}_{(n_k-1)\times(n_k-1)}. \tag{48}$$

By substituting (39),(40) and (41) to (38), it becomes

$$t(K^{\mathscr{F}}_{k,\overline{n}}) = n^{2k-2} \prod_{1\leqslant i\leqslant k} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & d_i+1 & 0 & 1 & \vdots \\ 1 & \ddots & \ddots & \ddots & 1 \\ \vdots & 1 & 0 & d_i+1 & 0 \\ 1 & \cdots & 1 & 0 & d_i \end{vmatrix}_{(n_i-1)\times(n_i-1)},$$

or

$$t(K^{\mathscr{F}}_{k,\overline{n}}) = n^{2k-2} \prod_{1\leqslant i\leqslant k} \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & d_i & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & d_i & -1 \\ 0 & \cdots & 0 & -1 & d_i-1 \end{vmatrix}_{(n_i-1)\times(n_i-1)}. \tag{49}$$

Thus, from (11.1) and (42) we have

$$t(K^{\mathscr{F}}_{k,\overline{n}}) = n^{2k-2} \prod_{1\leqslant i\leqslant k} \frac{\alpha_i^{n_i-1} - \beta_i^{n_i-1}}{\alpha_i - \beta_i} \tag{50}$$

where $\alpha_i = (d_i + \sqrt{d_i^2 - 4})/2$, $\beta_i = (d_i - \sqrt{d_i^2 - 4})/2$ and $d_i = n - n_i + 3$.

This is just the theorem. □

Corollary 4.2 *The total number of spanning trees of a fan graph F_n is*

$$t(F_n) = \frac{1}{\sqrt{5}}(\alpha^{n-1} - \beta^{n-1}) \tag{51}$$

where $\alpha = (3+\sqrt{5})/2$ and $\beta = (3-\sqrt{5})/2$.

Proof Let $k = 1$, then $n_1 = n$ and $K^{\mathscr{F}}_{1,\overline{n}} = F_n$. Substituting this fact into (35), this result is followed. □

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A Note on Antipodal Signed Graphs

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Abstract: A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ ($\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$) is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 1-signed graph or Smarandachely 1-marked graph is called abbreviated a *signed graph* or a *marked graph*. Singleton (1968) introduced the concept of the *antipodal graph* of a graph G, denoted by $A(G)$, is the graph on the same vertices as of G, two vertices being adjacent if the distance between them is equal to the diameter of G. Analogously, one can define the *antipodal signed graph* $A(S)$ of a signed graph $S = (G, \sigma)$ as a signed graph, $A(S) = (A(G), \sigma')$, where $A(G)$ is the underlying graph of $A(S)$, and for any edge $e = uv$ in $A(S)$, $\sigma'(e) = \mu(u)\mu(v)$, where for any $v \in V$, $\mu(v) = \prod\limits_{u \in N(v)} \sigma(uv)$. It is shown that for any signed graph S, its $A(S)$ is balanced and we offer a structural characterization of antipodal signed graphs. Further, we characterize signed graphs S for which $S \sim A(S)$ and $\overline{S} \sim A(S)$ where $\sim$ denotes switching equivalence and $A(S)$ and $\overline{S}$ are denotes the antipodal signed graph and complementary signed graph of S respectively.

Key Words: Smarandachely k-signed graphs, Smarandachely k-marked graphs, signed graphs, marked graphs, balance, switching, antipodal signed graphs, complement, negation.

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§1. Introduction

We consider only finite undirected graphs $G = (V, E)$ without loops and multiple edges and follow Harary [4] for notation and terminology.

A *Smarandachely k-signed graph* (*Smarandachely k-marked graph*) is an ordered pair $S = (G, \sigma)$ ($S = (G, \mu)$) where $G = (V, E)$ is a graph called *underlying graph of* S and $\sigma : E \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$ ($\mu : V \to (\overline{e}_1, \overline{e}_2, ..., \overline{e}_k)$) is a function, where each $\overline{e}_i \in \{+, -\}$. Particularly, a Smarandachely 1-signed graph or Smarandachely 1-marked graph is called abbreviated a *signed graph* or a *marked graph*.

In 1953, Harary published "On the notion of balance of a signed graph", [5], the first paper to introduce signed graphs. In this paper, Harary defined a signed graph as a graph whose edge

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set has been partitioned into positive and negative edges. He called a cycle positive if it had an even number of negative edges, and he called a signed graph balanced if every cycle is positive. Then he gave both necessary and sufficient conditions for balance.

Since then, mathematicians have written numerous papers on the topic of signed graphs. Many of these papers demonstrate the connection between signed graphs and different subjects: circuit design (Barahona [3], coding theory (Solé and Zaslavsky [20]), physics (Toulouse [22]) and social psychology (Abelson and Rosenberg [1]). While these subjects seem unrelated, balance plays an important role in each of these fields.

Four years after Harary's paper, Abelson and Rosenberg [1], wrote a paper in which they discuss algebraic methods to detect balance in a signed graphs. It was one of the first papers to propose a measure of imbalance, the "complexity" (which Harary called the "line index of balance"). Abelson and Rosenberg introduced an operation that changes a signed graph while preserving balance and they proved that this does not change the line index of imbalance. For more new notions on signed graphs refer the papers ([7]-[11], [13]-[19]).

A *marking* of S is a function $\mu : V(G) \to \{+, -\}$; A signed graph S together with a marking μ is denoted by S_μ. Given a signed graph S one can easily define a marking μ of S as follows: For any vertex $v \in V(S)$,

$$\mu(v) = \prod_{uv \in E(S)} \sigma(uv),$$

the marking μ of S is called *canonical marking* of S. In a signed graph $S = (G, \sigma)$, for any $A \subseteq E(G)$ the *sign* $\sigma(A)$ is the product of the signs on the edges of A.

The following characterization of balanced signed graphs is well known.

Proposition 1(E. Sampathkumar, [9]) *A signed graph $S = (G, \sigma)$ is balanced if, and only if, there exists a marking μ of its vertices such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$.*

Let $S = (G, \sigma)$ be a signed graph. Consider the marking μ on vertices of S defined as follows: each vertex $v \in V$, $\mu(v)$ is the product of the signs on the edges incident at v. *Complement* of S is a signed graph $\overline{S} = (\overline{G}, \sigma')$, where for any edge $e = uv \in \overline{G}$, $\sigma'(uv) = \mu(u)\mu(v)$. Clearly, $\overline{S}$ as defined here is a balanced signed graph due to Proposition 1.

The idea of switching a signed graph was introduced in [1]in connection with structural analysis of social behavior and also its deeper mathematical aspects, significance and connections may be found in [24].

Switching S with respect to a marking μ is the operation of changing the sign of every edge of S to its opposite whenever its end vertices are of opposite signs (See also R. Rangarajan and P. S. K. Reddy [8]). The signed graph obtained in this way is denoted by $\mathcal{S}_\mu(S)$ and is called *μ-switched signed graph* or just *switched signed graph*. Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *isomorphic*, written as $S_1 \cong S_2$ if there exists a graph isomorphism $f : G \to G'$ (that is a bijection $f : V(G) \to V(G')$ such that if uv is an edge in G then $f(u)f(v)$ is an edge in G') such that for any edge $e \in G$, $\sigma(e) = \sigma'(f(e))$. Further a signed graph $S_1 = (G, \sigma)$ *switches* to a signed graph $S_2 = (G', \sigma')$ (or that S_1 and S_2 are *switching equivalent*) written $S_1 \sim S_2$, whenever there exists a marking μ of S_1 such that $\mathcal{S}_\mu(S_1) \cong S_2$.

Note that $S_1 \sim S_2$ implies that $G \cong G'$, since the definition of switching does not involve change of adjacencies in the underlying graphs of the respective signed graphs.

Two signed graphs $S_1 = (G, \sigma)$ and $S_2 = (G', \sigma')$ are said to be *weakly isomorphic* (see [21]) or *cycle isomorphic* (see [23]) if there exists an isomorphism $\phi : G \to G'$ such that the sign of every cycle Z in S_1 equals to the sign of $\phi(Z)$ in S_2. The following result is well known (See [23]).

Proposition 2(T. Zaslavsky, [23]) *Two signed graphs S_1 and S_2 with the same underlying graph are switching equivalent if, and only if, they are cycle isomorphic.*

§2. Antipodal Signed Graphs

Singleton [11] has introduced the concept of antipodal graph of a graph G as the graph $A(G)$ having the same vertex set as that of G and two vertices are adjacent if they are at a distance of $diam(G)$ in G.

Motivated by the existing definition of complement of a signed graph, we extend the notion of antipodal graphs to signed graphs as follows: The *antipodal signed graph* $A(S)$ of a signed graph $S = (G, \sigma)$ is a signed graph whose underlying graph is $A(G)$ and sign of any edge uv is $A(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S. Further, a signed graph $S = (G, \sigma)$ is called antipodal signed graph, if $S \cong A(S')$ for some signed graph S'. The following result indicates the limitations of the notion $A(S)$ as introduced above, since the entire class of unbalanced signed graphs is forbidden to be antipodal signed graphs.

Proposition 3 *For any signed graph $S = (G, \sigma)$, its antipodal signed graph $A(S)$ is balanced.*

Proof Since sign of any edge uv in $A(S)$ is $\mu(u)\mu(v)$, where μ is the canonical marking of S, by Proposition 1, $A(S)$ is balanced. □

For any positive integer k, the k^{th} iterated antipodal signed graph $A(S)$ of S is defined as follows:

$$A^0(S) = S,\ A^k(S) = A(A^{k-1}(S))$$

Corollary 4 *For any signed graph $S = (G, \sigma)$ and any positive integer k, $A^k(S)$ is balanced.*

In [2], the authors characterized those graphs that are isomorphic to their antipodal graphs.

Proposition 5(Aravamudhan and Rajendran, [2]) *For a graph $G = (V, E)$, $G \cong A(G)$ if, and only if, G is complete.*

We now characterize the signed graphs that are switching equivalent to their antipodal signed graphs.

Proposition 6 *For any signed graph $S = (G, \sigma)$, $S \sim A(S)$ if, and only if, $G = K_p$ and S is balanced signed graph.*

Proof Suppose $S \sim A(S)$. This implies, $G \cong A(G)$ and hence G is K_p. Now, if S is any signed graph with underlying graph as K_p, Proposition 3 implies that $A(S)$ is balanced and hence if S is unbalanced and its $A(S)$ being balanced can not be switching equivalent to S in accordance with Proposition 2. Therefore, S must be balanced.

Conversely, suppose that S is a balanced signed graph and G is K_p. Then, since $A(S)$ is balanced as per Proposition 3 and since $G \cong A(G)$, this follows from Proposition 2 again. □

Proposition 7 *For any two signed graphs S and S' with the same underlying graph, their antipodal signed graphs are switching equivalent.*

Proposition 8(Aravamudhan and Rajendran, [2]) *For a graph $G = (V, E)$, $\overline{G} \cong A(G)$ if, and only if, i). G is diameter 2 or ii). G is disconnected and the components of G are complete graphs.*

In view of the above, we have the following result for signed graphs:

Proposition 9 *For any signed graph $S = (G, \sigma)$, $\overline{S} \sim A(S)$ if, and only if, G satisfies conditions of Proposition 8.*

Proof Suppose that $A(S) \sim \overline{S}$. Then clearly we have $A(G) \cong \overline{G}$ and hence G satisfies conditions of Proposition 8.

Conversely, suppose that G satisfies conditions of Proposition 8. Then $\overline{G} \cong A(G)$ by Proposition 8. Now, if S is a signed graph with underlying graph satisfies conditions of Proposition 8, by definition of complementary signed graph and Proposition 3, $\overline{S}$ and $A(S)$ are balanced and hence, the result follows from Proposition 2. □

The notion of *negation* $\eta(S)$ of a given signed graph S defined in [6] as follows: $\eta(S)$ *has the same underlying graph as that of S with the sign of each edge opposite to that given to it in S*. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in S while applying the unary operator $\eta(.)$ of taking the negation of S.

Propositions 6 and 9 provides easy solutions to two other signed graph switching equivalence relations, which are given in the following results.

Corollary 10 *For any signed graph $S = (G, \sigma)$, $S \sim A(\eta(S))$.*

Corollary 11 *For any signed graph $S = (G, \sigma)$, $\overline{S} \sim A(\eta(S))$.*

Problem 12 *Characterize signed graphs for which i) $\eta(S) \sim A(S)$ or ii) $\eta(\overline{S}) \sim A(S)$.*

For a signed graph $S = (G, \sigma)$, the $A(S)$ is balanced (Proposition 3). We now examine, the conditions under which negation $\eta(S)$ of $A(S)$ is balanced.

Proposition 13 *Let $S = (G, \sigma)$ be a signed graph. If $A(G)$ is bipartite then $\eta(A(S))$ is balanced.*

Proof Since, by Proposition 3, $A(S)$ is balanced, if each cycle C in $A(S)$ contains even

number of negative edges. Also, since $A(G)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $A(S)$ is also even. Hence $\eta(A(S))$ is balanced. $\square$

§3. Characterization of Antipodal Signed Graphs

The following result characterize signed graphs which are antipodal signed graphs.

Proposition 14 *A signed graph $S = (G, \sigma)$ is an antipodal signed graph if, and only if, S is balanced signed graph and its underlying graph G is an antipodal graph.*

Proof Suppose that S is balanced and G is a $A(G)$. Then there exists a graph H such that $A(H) \cong G$. Since S is balanced, by Proposition 1, there exists a marking μ of G such that each edge uv in S satisfies $\sigma(uv) = \mu(u)\mu(v)$. Now consider the signed graph $S' = (H, \sigma')$, where for any edge e in H, $\sigma'(e)$ is the marking of the corresponding vertex in G. Then clearly, $A(S') \cong S$. Hence S is an antipodal signed graph.

Conversely, suppose that $S = (G, \sigma)$ is an antipodal signed graph. Then there exists a signed graph $S' = (H, \sigma')$ such that $A(S') \cong S$. Hence G is the $A(G)$ of H and by Proposition 3, S is balanced. $\square$

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Group Connectivity of 1-Edge Deletable IM-Extendable Graphs

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Abstract: A graph G is called a k-edges deletable IM-extendable graph, if $G-F$ is IM-extendable for every $F \subseteq E(G)$ with $|F|=k$. Denoted by $\wedge_g(G)$ the group connectivity of a graph G. In this paper, $\wedge_g(G)=3$ is gotten if G is a 4-regular claw-free 1-edge deletable IM-extendable graph.

Key Words: Graph, multi-group connectivity, group connectivity, induced matching.

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§1. Introduction and Lemmas

In 1950s, Tutte introduced the theory of nowhere-zero flows as a tool to investigate the coloring problem of maps, together with his most fascinating conjectures on nowhere-zero flows. These have been extended by Jaeger, Linial, Payan and Tarsil in 1992 to group connectivity, the generalized form of nowhere-zero flows. Let G be an undirected graph and $\widetilde{A}=(\bigcup\limits_{i=1}^{m} A_i;\{+_i, 1 \leq i \leq m\})$ be an Abelian multi-group. Let $\widetilde{A}^*$ denote the set of non-zero elements of $\widetilde{A}$. A function $b: V(G) \to \widetilde{A}$ is called an *$\widetilde{A}$-valued zero-sum function* of G if $\sum_{v\in V(G)} b(v)=0_{+_i}$, $1 \leq i \leq m$ in G. The set of all $\widetilde{A}$-valued zero-sum function on G is denoted by $Z(G,\widetilde{A})$. We define: $F(G,\widetilde{A})=\{f: E(G)\to \widetilde{A}\}$ and $F^*(G,\widetilde{A})=\{f: E(G)\to \widetilde{A}^*\}$. Let G^1 be an orientation of a graph G. If an edge $e\in E(G)$ is directed from a vertex u to a vertex v, then let tail$(e)=u$ and head$(e)=v$. For a vertex $v\in V(G)$, let $E^-(v)=\{e\in E(G^1): v=tail(e)\}$, and $E^+(v)=\{e\in E(G^1): v=head(e)\}$. Given a function $f\in F(G,\widetilde{A})$, define $\partial f: V(G)\to A$ by $\partial f(v)=\sum\limits_{e\in E^+(v)} f(e)-\sum\limits_{e\in E^-(v)} f(e)$. A graph G is *$\widetilde{A}$-connected* if G has an orientation G^1 such that for every function $b\in Z(G,\widetilde{A})$, there is a function $f\in F^*(G^1,\widetilde{A})$ such that $b=\partial f$. Let $\langle\widetilde{A}\rangle$ be the family of graphs that are $\widetilde{A}$-connected. The *multi-group connectivity* of G is defined as: $\wedge_g(G)$=min{k | if $\widetilde{A}$ is an Abelian group with $|\widetilde{A}|\geq k$, then $G\in\langle\widetilde{A}\rangle$}. Particularly,

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if $m = 1$, i.e., $\widetilde{A} = (A, +)$ an Abelian group, such connectivity is called *group connectivity.*

Let v be a vertex of G, denote $N(v)$ *by* : $N(v) = \{u \in V(G) - v : uv \in E(G)\}$. Let u be a vertex of G, denote $N^2(u) = N(N(u))\backslash(N(u) \cup \{u\})$. Graph G is called *claw-free*, if it doesn't contain $k_{1,3}$ as an induced subgraph. Let C_n^k denote the graph with $V(C_n^k) = V(C_n)$, $E(C_n^k) = \{uv : u, v \in V(C_n) \text{ and } d_{c_n}(u, v) \leq k\}$, where $d_{c_n}(u, v)$ is a distance between u and v in C_n. Let G be a graph. A *triangle-path* in G is a sequence of distinct triangles $T_1\ T_2 \cdots T_m$ in G such that for $1 \leq i \leq m-1$, the following formula (*) holds:

$$|E(T_i) \cap E(T_{i+1})| = 1 \quad \text{and} \quad E(T_i) \cap E(T_j) = \Omega \ \text{ if } j > i+1. \qquad (*)$$

Furthermore, if $m \geq 3$ and (*) holds for all i, $1 \leq i \leq m$, with the additionally taken mod m, then the sequence is called a *triangle-cycle.* The number m is the length of the triangle-path(triangle-cycle). A connected graph G is *triangularly connected* if for any distinct $e, e_1 \in E(G)$, which are not parallel, there is a triangle-path $T_1T_2 \cdots T_m$ such that $e \in E(T_1)$ and $e_1 \in E(T_m)$.

Let G be a connected graph, $V(G)$ and $E(G)$ denote its sets of vertices and edges, respectively. For $S \subseteq V(G)$, let $E(S) = \{uv \in E(G), u, v \in S\}$. For $M \subseteq E(G)$, let $V(M) = \{u \in V(G)$: there is $v \in V(G)$ such that $uv \in M\}$. A set of edges $M \subseteq E(G)$ is called a *matching* of G if they are independent in G, and no two of them share a common end vertex. A matching is called *perfect* if it covers all vertices of G. A matching M is called induced matching if $E(V(M)) = M$. G is called *induced matching extendable* if every induced matching M of G is contained in a perfect matching of G. For simplicity, induced matching extendable will often be abbreviated as *IM-extendable.*

Notations undefined in this paper will follow [1]. In this paper, we give some properties of the 4-regular claw-free 1-edge deletable IM-extendable and prove that its group connectivity is 3.

Lemma 1.1.([1]) *A graph G has a perfect matching if and only if for every* $S \subseteq V(G)$, $o(G-S) \leq |S|$, *Where* $o(H)$ *is the number of odd components of* H.

For group connectivity, some conclusions have reached. For example, complete graphs, complete bipartite graphs and triangularly connected graphs etc in [2,3,4,5,6]. For more results about IM-extendable graphs, one can see references [7,8,9,10].

A *k-circuit* is a circuit of k vertices. A *wheel* W_k is the graph obtained from a k-circuit by adding a new vertex, called the center of the wheel, which is joined to every vertex of the k-circuit. W_k is an odd(even) wheel if k is odd(even). For a technical reason, a single edge is regarded as 1-circuit, and thus W_1 is a triangle, called the trivial wheel.

Lemma 1.2([6]) (1)$W_{2n} \in \langle Z_3 \rangle$.
(2) *Let* $G \cong W_{2n+1}$, $b \in Z(G, Z_3)$. *Then there exists a* (Z_3, b)-NZF $f \in F^*(G, Z_3)$ *if and only if* $b \neq 0$.

Lemma 1.3([4]) *Let G be a connected graph with n vertices and m edges. Then* $\wedge_g(G)$=2 *if and only if* $n = 1$ (*and so G has m loops*).

Lemma 1.4([3]) *Let* $H \leq G$ *be* Z_k*-connected. If* G/H *is* Z_k*-connected, then so is* G.

§2. Main Results

Lemma 2.1 *C_6^2 is 4-regular claw-free 1-edge deletable connected IM-extendable graph.*

Proof Obviously, C_6^2 is 4-regular and claw-free. The following we will prove C_6^2 is 1-edge deletable IM-extendable graph. Supposing the vertices of C_6^2 denoted by v_i, $1 \le i \le 6$, along a clockwise. Supposing M is an induced matching of an induced graph $G[N(u)]$. Since $G[N(u)]$ has four vertices, so $|M| \le 2$. If $|M| = 2$, u is an isolated vertex of $G - V(M)$, this conflict with that G is 1-edge deletable IM-extendable graph, thus, $|M| \le 1$. Since $E(C_6^2) = E_1 \cup E_2$ where $E_1 = E(C_6)$, $E_2 = E \setminus E_1$, the following discussions are divided into two cases.

Case 1 Deleting one edge in E_1. Without loss of generality, suppose deleting edge v_1v_2. From the structure of C_6^2, if M is an induced matching of C_6^2, then $|M| < 2$. If let $M = v_2v_3$ be an induced matching of $G - \{v_1v_2\}$, it extended to a perfect matching $\{v_2v_3, v_4v_5, v_6v_1\}$. Otherwise, let $M = \{v_2v_4\}$ be an induced matching of $G - \{v_1v_2\}$, it extended to a perfect matching $\{v_2v_4, v_3v_5, v_6v_1\}$.

Case2 Deleting one edge in E_2. Without loss of generality, let $M = \{v_2v_3\}$ be an induced matching of $G - \{v_1v_3\}$, it extended to a perfect matching $\{v_2v_3, v_4v_5, v_6v_1\}$. Otherwise, let $M = \{v_3v_5\}$ be an induced matching of $G - \{v_1v_3\}$, it extended to a perfect matching $\{v_3v_5, v_2v_4, v_6v_1\}$. So C_6^2 is a 4-regular claw-free 1-edge deletable connected IM-extendable graph. □

Lemma 2.2 *Let G be a 4-regular claw-free 1-edge deletable connected IM-extendable graph, then $G = C_6^2$.*

Proof For a given $u \in V(G)$, since G is 4-regular, let $N(u) = \{x_1, x_2, x_3, x_4\}$, $N^2(u) = N(N(u)) \setminus (N(u) \cup \{u\})$, $f(u) = |E(N(u))|$, $g(N(u)) = |N^2(u)|$.

Since G is 4-regular and claw-free, we have $2 \le f(u) \le 6$. If $f(u) = 6$, G is isomorphic to K_5, there is no perfect matching in K_5 which is a contradiction with 1-edge deletable IM-extendable graph. If $f(u) = 5$, $G[N(u)]$ is isomorphic to $K_4 - e$. Without loss of generality, let $e = x_2x_4$ and $y_1 \in N(x_2) \setminus (\{u\} \cup N(u))$. Since G is 4-regular, there exists $z \in N(y_1) \setminus N(u)$. Let $M = \{y_1z, ux_1\}$ be an induced matching of $G - \{x_2x_3\}$. It could not extend to a perfect matching of $G - \{x_2x_3\}$ which is a contradiction. Next we discuss: $2 \le f(u) \le 4$. There are three cases according to the value of $f(u)$.

Case 1 $f(u) = 2$.

Let x_1, x_2, x_3, x_4 be neighbor vertices of u, we have three different subcases: Subcase(a) $x_1x_2, x_2x_3 \in E(G)$; Subcase(b) $x_1x_2, x_3x_4 \in E(G)$; Subcase(c) $x_1x_2, x_1x_3 \in E(G)$.

For subcases (a), the induced subgraph of G by vertices u, x_1, x_3, x_4 consists of an induced subgraph $k_{1,3}$ which contradict with the assumption. For subcase (b), $M = \{x_1x_2, x_3x_4\}$ is an induced matching of $G - \{ux_2\}$, however, u could not included in vertex set of any perfect matching of $G - \{ux_2\}$ which contradict with the assumption. For subcase (c), the induced subgraph of G by vertices u, x_2, x_3, x_4 consists of $k_{1,3}$, which contradicts with the assumption. Therefore $f(u) \ne 2$.

Case 2 $f(u) = 3$.

$G[N(u)]$ is isomorphic to P_4 or $K_3 \cup K_1$ or $K_{1,3}$, where P_n is the path with n vertices.

If $G[N(u)]$ is isomorphic to P_4. Supposing $x_1x_2, x_2x_3, x_3x_4 \in E(G)$, then $M = \{x_1x_2, x_3x_4\}$ is an induced matching of $G - \{x_2x_3\}$, but it does not extended to a perfect matching of $G - \{x_2x_3\}$.

If $G[N(u)]$ is isomorphic to $k_{1,3}$, obviously G consists of $k_{1,3}$ as its induced subgraph, contradiction.

If $G[N(u)]$ is isomorphic to $K_3 \cup K_1$, supposing $x_1x_2, x_2x_3, x_1x_3 \in E(G)$, then $3 \leq g(N(u)) \leq 6$, there are four subcases according to the value of $g(N(u))$.

Subcase 1. If $g(N(u)) = 3$, let $N^2(u) = \{y_1, y_2, y_3\}$ then $y_ix_4 \in E(G)$ $(i = 1, 2, 3)$. Since G is claw-free and 4-regular, then $y_1y_2, y_1y_3, y_2y_3 \in E(G)$ and each vertex of x_i is adjacent to only one vertices of y_j (where $i, j \in \{1, 2, 3\}$ are distinct). Without loss of generality, let x_1, x_2, x_3 are adjacent to y_1, y_2, y_3 respectively. Let $M = \{x_1x_3, x_4y_2\}$ be an induced matching of $G - \{ux_2\}$. It does not extended to a perfect matching of $G - \{ux_2\}$ which is a contradiction.

Subcase 2. If $g(N(u)) = 4$, let $y_i \in N^2(u), i = 1, 2, 3, 4$. Since $d(x_4) = 4$, there is three of $y_i (1 \leq i \leq 4)$ adjacent to x_4, without loss of generality, supposing $y_ix_4 \in E(G)$ $(i = 2, 3, 4)$. Because G is claw-free, one have $y_2y_3, y_3y_4, y_2y_4 \in E(G)$. Obviously y_1 is adjacent to at least one vertex of x_i (i=1,2,3).

If y_1 is adjacent to each of x_i (i=1,2,3), that is $x_1y_1, x_2y_1, x_3y_1 \in E(G)$. If $y_1y_4 \notin E(G)$, let $M = \{x_1y_1, x_4y_4\}$ be an induced matching of $G - \{x_2x_3\}$ which could not extended to a perfect matching of $G - \{x_2x_3\}$. If $y_1y_4 \in E(G)$, let $M = \{x_1y_1, x_4y_3\}$ be an induced matching of $G - \{x_2x_3\}$ which could not extended to a perfect matching of $G - \{x_2x_3\}$.

If y_1 is adjacent to two vertices of x_i (i=1,2,3), without loss of generality, let $x_1y_1, x_2y_1 \in E(G)$ and $x_3y_1 \notin E(G)$, x_3 is adjacent to one of y_i (which $i \in \{2, 3, 4\}$). If $x_3y_3 \notin E(G)$, let $M = \{x_2x_3, x_4y_3\}$ be an induced matching of $G - \{y_2y_4\}$ which could not extended to a perfect matching of $G - \{y_2y_4\}$. If $x_3y_3 \in E(G)$, let $M = \{x_2x_3, x_4y_4\}$ be an induced matching of $G - \{ux_1\}$ which could not extended to a perfect matching of $G - \{ux_1\}$.

If y_1 is adjacent to only one of x_i (i=1,2,3), without loss of generality, let $x_1y_1 \in E(G)$. Supposing $x_2y_2, x_3y_3 \in E(G)$, let $M = \{x_1x_2, x_4y_3\}$ be an induced matching of $G - \{ux_3\}$ which could not extended to a perfect matching of $G - \{ux_3\}$.

Subcase 3. If $g(N(u)) = 5$, let $N^2(u) = \{y_1, y_2, y_3, y_4, y_5\}$ and supposing $y_3x_4, y_4x_4, y_5x_4 \in E(G)$. Because G is claw-free, $y_3y_4, y_4y_5, y_5y_3 \in E(G)$. Since no more than two vertices of x_i $(i = 1, 2, 3)$ is adjacent to y_j $(j = 1, 2)$, without loss of generality, let $x_1y_1, x_2y_2 \in E(G)$.

If x_3 is adjacent to y_1 or y_2, supposing $x_3y_1 \in E(G)$. Let $M = \{x_1x_3, x_4y_3\}$ be an induced matching of $G - \{ux_2\}$. It does not extended to a perfect matching of $G - \{ux_2\}$ which is a contradiction.

If $x_3y_i \notin E(G)$ (i=1,2). Without loss of generality, supposing $x_3y_3 \in E(G)$. Let $M = \{x_1x_3, x_4y_5\}$ be an induced matching of $G - \{ux_2\}$ which could not extended to a perfect matching of $G - \{ux_2\}$.

Subcase 4. If $g(N(u)) = 6$, without loss of generality, supposing $y_ix_4 \in E(G)$ $(i = 4, 5, 6)$.

Because each vertex of x_i (i=1,2,3) has degree 3 in $N(u)$, each of x_i $(i = 1, 2, 3)$ is adjacent to only one of y_j $(j = 1, 2, 3)$, without loss of generality, let $x_iy_i \in E(G)$ $(i = 1, 2, 3)$. Let $M = \{x_1x_3, x_4y_4\}$ be an induced matching of $G - \{ux_2\}$ which could not extended to a perfect matching of $G - \{ux_2\}$. So $f(u) \neq 3$.

Case 3 If $f(u) = 4$, $G[N(u)]$ is isomorphic to C_4 or $K_{1,3} + e$.

If $G[N(u)]$ is isomorphic to C_4. Since there are only four edges between $N(u)$ and $N^2(u)$, then $1 \leq g(N(u)) \leq 4$.

(3.1) If $g(N(u)) = 1$. Supposing $v \in N^2(u)$, there exists $x_iv \in E(G)$ $(i = 1, 2, 3, 4)$, it is isomorphic to C_6^2.

(3.2) If $g(N(u)) = 2$. Supposing $y_1, y_2 \in N^2(u)$. There are two subcases. Subcases 1, there are two vertices adjacent to y_1, two vertices adjacent to y_2. Subcases 2, there are three vertices adjacent to y_1, only one vertex adjacent to y_2.

Subcase 1. Supposing $x_1y_1, x_2y_1, x_3y_2, x_4y_2 \in E(G)$, there exists z, satisfying $y_1z \in E(G)$. Let $M = \{x_3x_4, y_1z\}$ be an induced matching of $G - \{x_1x_2\}$ which could not extended to a perfect matching of $G - \{x_1x_2\}$.

Subcase 2. Supposing $x_1y_1, x_2y_1, x_3y_1, x_4y_2 \in E(G)$. However, x_4, x_3, x_1, y_2 induced a $K_{1,3}$ which is a contradiction.

(3.3) If $g(N(u)) = 3$. Supposing $y_1, y_2, y_3 \in N^2(u)$. Obviously, only two vertices of x_i (i=1,2,3,4) are adjacent to one of y_i (i=1,2,3). Without loss of generality, let $x_iy_i, x_4y_3 \in E(G)$ $(i = 1, 2, 3)$. x_2, x_1, y_2, x_3 induced a $K_{1,3}$ which is a contradiction.

(3.4) If $g(N(u)) = 4$. Supposing $y_1, y_2, y_3, y_4 \in N^2(u)$. Without loss of generality, let $x_iy_i \in E(G)$ $(i = 1, 2, 3, 4)$. x_2, x_1, y_1, x_4 induced a $K_{1,3}$ subgraph. If $G[N(u)]$ is isomorphic to C_4, it does not extended to a perfect matching. If $G[N(u)]$ is isomorphic to $K_{1,3} + e$, supposing $x_1x_2, x_1x_3, x_1x_4, x_3x_4 \in E(G)$, since there are only 4 edges between $N(u)$ and $N^2(u)$, one have $2 \leq g(N(u)) \leq 4$. There are three subcases according to the value of $g(N(u))$.

Subcase 1. If $g(N(u)) = 2$. Supposing $N^2(u) = \{y_1, y_2\}$, then $x_2y_1, x_2y_2 \in E(G)$. Because G is claw-free, $y_1y_2 \in E(G)$. If both of x_3, x_4 are adjacent to y_1, let $M = \{ux_1, y_1y_2\}$ be an induced matching of $G - \{x_3x_4\}$ which could not extended to a perfect matching of $G - \{x_3x_4\}$. If $x_3y_1, x_4y_2 \in E(G)$, let $M = \{ux_1, y_1y_2\}$ be an induced matching of $G - \{x_3x_4\}$ which could not extended to a perfect matching of $G - \{x_3x_4\}$.

Subcase 2. If $g(N(u)) = 3$, supposing $N^2(u) = \{y_1, y_2, y_3\}$. Without loss of generality, let $x_2y_1, x_2y_2 \in E(G)$. Because G is claw-free, $y_1y_2 \in E(G)$. If both of x_3, x_4 are adjacent to y_3, let $M = \{y_1y_2, x_3x_4\}$ be an induced matching of $G - \{x_1x_3\}$. However, not all vertices of x_1, x_2, u could be included in a perfect matching vertices of $G - \{x_1x_3\}$ which is a contradiction. If $x_3y_3, x_4y_2 \in E(G)$, let $M = \{x_1x_3, y_1y_2\}$ be an induced matching of $G - \{x_1x_2\}$. However, not all vertices of x_2, x_4, u could be included in a vertex set of perfect matching of $G - \{x_1x_1\}$ which is a contradiction.

Subcase 3. If $g(N(u)) = 4$, Supposing $N^2(u) = \{y_1, y_2, y_3, y_4\}$. Without loss of generality,

let $x_2y_1, x_2y_2, x_3y_3, x_4y_4 \in E(G)$. Because G is claw-free, $y_1y_2 \in E(G)$. Let $M = \{x_1x_4, y_1y_2 \in E(G)\}$ be an induced matching of $G - \{ux_2\}$ which could not extended to a perfect matching of $G - \{ux_2\}$. The lemma 2.2 is proved. □

Lemma 2.3 *If* $n \geq 5$, *then* C_n^2 *is* Z_3*-connected.*

Proof By the definition of C_n^k, for $n \geq 5$, there exists a subgraph of C_n^2 isomorphic to W_{2k}. By lemma 1.2, W_{2k} is Z_3-connected. Since C_n^2 is triangularly connected, by contracting W_{2k} in C_n^2 and Lemma 1.4, we have C_n^2 is Z_3-connected. □

Theorem 2.4 *The group connectivity of* 4*-regular claw-free* 1*-edge deletable* IM*-extendable graph is* 3.

Proof By applying lemma 2.1 and lemma2.2, 4-regular claw-free 1-edge deletable IM-extendable graph is C_6^2. From Lemma 2.3, the group connectivity of 4-regular claw-free 1-edge deletable IM-extendable graph is not more than 3. By lemma 1.3 we conclude the group connectivity of 4-regular claw-free 1-edge deletable IM-extendable graph is more than 2. Therefore, the group connectivity of 4-regular claw-free 1-edge deletable IM-extendable graph is 3. This theorem is proved. □

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A Note On Line Graphs

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Abstract: In this note we define two generalizations of the line graph and obtain some results. Also, we mark some open problems.

Key Words: Smarandachely (k, r) line graph, (k, r)-graphs, line graphs

AMS(2010): 05C75, 05C45

For standard terminology and notion in graph theory we refer the reader to Harary [2]; the non-standard will be given in this paper as and when required. We treat only finite simple graphs without self loops and isolates.

The line graph $L(G)$ of a graph G is defined to have as its vertices the edges of G, with two being adjacent if the corresponding edges share a vertex in G. Line graphs have a rich history. The name line graph was first used by Harary and Norman [3] in 1960. But line graphs were the subject of investigation as far back as 1932 in Whitney s paper [7], where he studied edge isomorphism and showed that for connected graphs, edge-isomorphism implies isomorphism except for K_3 and $K_{1,3}$. The first characterization (partition into complete subgraphs) was given by Krausz [5]. Instead, we refer the interested reader to a somewhat older but still an excellent survey on line graphs and line digraphs by Hemminger and Beineke [4]. An excellent book by Prisner [6] describes many interesting generalizations of line graphs. In this note we generalize the line graph $L(G)$ of G as follows:

Let $G = (V, E)$ be a graph of order $p \geq 3$, k and r be integers with $1 \leq r < k \leq p$. Let $U = \{S_1, S_2, ..., S_n\}$ be the set of all distinct connected acyclic subgraphs of G of order k and $U' = \{T_1, T_2, ..., T_m\}$ be the set of all distinct connected subgraphs of G with size k.

The *vertex* (k, r)*-graph* $L^v_{(k,r)}(G)$, where $1 \leq r < k \leq p$ is the graph has the vertex set U where two vertices S_i and S_j, $i \neq j$ are adjacent if, and only if, $S_i \cap S_j$ has a connected subgraph of order r.

The *edge* (k, r)*-graph* $L^e_{(k,r)}(G)$, where $0 \leq r < k \leq q$ is the graph has the vertex set U' where two vertices T_i and T_j, $i \neq j$ are adjacent if, and only if, $T_i \cap T_j$ has a connected subgraph of size r.

The *Smarandachely* (k, r) *line graph* $L^S_{(k,r)}(G)$ of a graph G is such a graph with vertex set U' and two vertices T_i and T_j, $i \neq j$ are adjacent if and only if, $T_i \cap T_j$ has a connected subgraph with order or size r. Clearly, $L^v_{(k,r)}(G) \leq L^S_{(k,r)}(G)$ and $L^e_{(k,r)}(G) \leq L^S_{(k,r)}(G)$. In

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Figure 1, we depicted $L(G)$, $L^v_{(k,r)}(G)$ and $L^e_{(k,r)}(G)$ for the graph G.

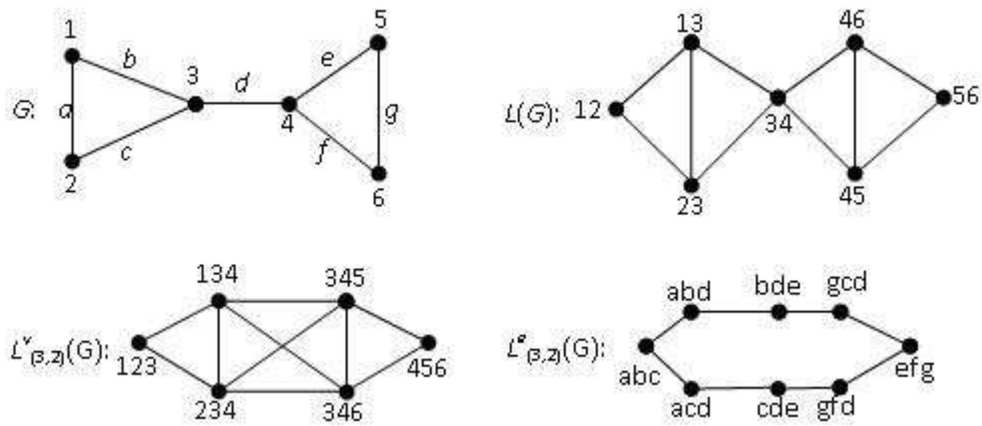

Figure.1

One can easily verify that: $L_3(K_{1,3}) = K_3$, $L_3(C_n) = C_n$, for $n \geq 3$, $L_3(P_n) = P_{n-1}$ and $L_3(K_4) = L(K_4 - e) = K_4$.

For any positive integer k, the k^{th} iterated line graph $L(G)$ of G is defined as follows: $L^0(G) = G$, $L^k(G) = L(L^{k-1}(G))$.

A graph G is a $(3,2)$-*graph* if there exists a graph H such that $L_{(3,2)}(H) \cong G$. First we prove the following result:

Proposition 1 *For any graph G, $L_{(3,2)} \cong L^2(G)$.*

Proof First we show that $L_{(3,2)(G)}$ and $L^2(G)$ have the same number of vertices. Let S be a vertex in $L_{(3,2)}(G)$. Then S corresponds to a subgraph of order 3 in G. Say, S consists of two adjacent edges ab and bc. Then corresponding to S we have an edge in $L(G)$ with end vertices ab and bc, and corresponding this edge, we have a vertex say abc in $L^2(G)$. Similarly, we can show that very vertex in $L^2(G)$ corresponds to a connected subgraph of order 3. This proves that $L_{(3,2)}(G)$ and $L^2(G)$ have the same number of vertices. Now, let S_1 and S_2 be two adjacent edges in $L_{(3,2)}(G)$. Then S_1 and S_2 correspond to two connected subgraphs of order 3 each, having a common edge. These in turn will give two adjacent edges, say $e(S_1)$ and $e(S_2)$ in $L(G)$ and this will give an edge in $L^2(G)$ with end vertices $e(S_1)$ and $e(S_2)$. This proves the result. $\Box$

In general one can establish the following result.

Proposition 2 *Let G be a graph of order p and $2 \geq r < k \leq p$. If $\triangle(G) \leq 2$ then $L_{(k,k-1)}(G) \cong L^{k-1}(G)$.*

Note that this is true only when $\triangle(G) \leq 2$. For example, we find $L^{n-1}_{(n,n-1)}(K_{1,n}) = K_n = L(K_{1,n})$.

Proposition 3 *The star $K_{1,3}$ is not a 3-line graph.*

Proof Let $K_{1,3}$ be a 3-line graph. Then there exists a graph G such that $L(K_{1,3}) = G$. Since $K_{1,3}$ has four vertices, G should have exactly four connected subgraphs each of order three. All connected graphs having exactly four induced subgraphs are as follows: i) C_4, ii) K_4, iii) $K_4 - e$ and iv) P_6. None of these graphs have $K_{1,3}$ as its 3-line graph. □

Clearly, any graph having $K_{1,3}$ as an induced subgraph is not a 3-line graph. Hence, we have

Corollary 4 *$K_{1,n}$, $n \geq 3$ is not a 3-line graph.*

It is not true in general that the line graph $L(G)$ of a graph G is a subgraph of $L_3(G)$. For example, $L_3(K_{1,4})$ does not contain K_4, the line graph of $K_{1,4}$ as a subgraph.

Problem 5 *Characterize 3-line graphs.*

A graph G is a *self 3-line graph*, if it is isomorphic to its 3-line graph.

Problem 6 *Characterize self 3-line graphs.*

Proposition 7 *Let $L_3(G)$ be the 3-line graph of a graph G of order $p \geq 3$. The degree of a vertex s in $L_3(G)$ is denoted by $degs$ and is defined as follows:*

Let S be the subgraph of G corresponding to the vertex s in $L_3(G)$. For an edge $x = uv$ in S, let $d(x) = (deg_G u + deg_G v) - (deg_S u + deg_S v)$, where $deg_G u$ and $deg_S u$ are the degrees of u in G and S respectively. Then $d(s) = \sum\limits_{x \in S} d(x)$.

Proof Consider an edge uv in S. Suppose $y = uz$ is an edge of G at u which is not in S. Then y belongs to a connected subgraph S_1 of cardinality three containing the edge uv which is distinct form S. Since S and S_1 have common edge, ss_1 is an edge in $L_3(G)$, where s_1 is the vertex in $L_3(G)$ corresponding to the subgraph S_1 in G. Similarly, for any edge $y_1 = vz_1$ at v in G which is not in S, we have an edge ss_2 in $L_3(G)$. This implies that corresponding to the edge $x = uv$ in S, we have $(deg_G u - deg_S u) + (deg_G v - deg_S v)$ edges in $L_3(G)$, and hence the result follows. □

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One-Mother Vertex Graphs

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Abstract: In this paper we will define a new type of graph. The idea of this definition is based on when we illustrate the cardiovascular system by a graph we find that not all vertices have the same important so we define this new graph and call it 1- mother vertex graph.

Key Words: Smarandache mother-father graph, 1-mother graph, matrices

AMS(2010): 97K30, 94C15

§1. Introduction

Unlike other areas in mathematics, graph theory traces its beginning to definite time and place: the problem of the seven bridges of Königsberg, which was solved in 1736 by Leonhard Euler. And in 1752 we find Euler's Theorem for planer graph. However, after this development, little was accomplished in this area for almost a century [4].

here are many physical systems whose performance depends not only on the characteristics of the components but also on the relative locations of the elements. An obvious example is an electrical network. One simple way of displaying a structure of a system is to draw a diagram consisting of points called vertices and line segments called edges which connect these vertices so that such vertices and edges indicate components and relationships between these components. Such a diagram is called linear graph. A graph G is a triple consisting of a vertex-set V(G), an edge-set E(G) and a relation that associated with each edge two vertices called its endpoints.

§2. Definitions and Background

Definition 2.1 *An abstract graph G is a diagram consisting of finite non empty set of elements called vertices denoted by $V(G)$ together with a set of unordered pairs of these elements called edges denoted by $E(G)$. The set of vertices of the graph G is called the vertex-set of G and the list of the edges is called the edge-list of G [1,5,9,10].*

Definition 2.2 *An oriented abstract graph is a pair (V,E) where V is finite non empty set of vertices and E is a set of ordered pairs of distinct elements of E with the property that if $(v,w)\in E$ then $(w,v)\notin E$ where the element (v,w) denote the edge from v to w [4,5].*

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Definition 2.3 *An empty graph is a graph with no vertices and no edges [5].*

Definition 2.4 *A null graph is a graph containing no edges [9,10].*

Definition 2.5 *A multiple edges defined as two or more edges joining the same pair of vertices [1,8,9,10].*

Definition 2.6 *A loop is an edge joining a vertex to itself [1,8,9,10].*

Definition 2.7 *A simple graph is a graph with no loops or multiple edges [9].*

Definition 2.8 *A multiple graph is a graph with allows multiple edges and loops [1,8,9,10].*

Definition 2.9 *A complete graph is a graph in which every two distinct vertices are joined by exactly one edge [5,6,9,10].*

Definition 2.10 *A connected graph is a graph that in one piece, where as one which splits in to several pieces is disconnected [9].*

Definition 2.11 *Given a graph G, a graph H is called a subgraph of G if the vertices of H are vertices of G and the edges of H are edges of G [5,6,8].*

Definition 2.12 *Let v and w be two vertices of a graph. If v and w are joined by an edges, then v and w are said to be adjacent. Also, v and w are said to be incident with e then e is said to be incident with v and w [10].*

Definition 2.13 *Let G be a graph without loops, with n-vertices labeled $1, 2, 3, ..., n$. The adjacency matrix $A(G)$ is the $n \times n$ matrix in which the entry in row i and column j is the number of edges joining the vertices i and j [10].*

Definition 2.14 *Let G be a graph without loops, with n- vertices labeled 1,2,3,..., n and m edges labeled 1,2,3,...,m. The incidence matrix $I(G)$ is the $n{\times}m$ matrix in which the entry in row i and column j is l if vertex i is incident with edge j and 0 otherwise [10].*

§3. Main Results

In this article, we will define new types of graphs as follows:

Definition 3.1 *A Smarandache mother-father graph is a graph G in which there are vertices $u_m^1, u_m^2, \cdots, u_m^n$, $v_m^1, v_m^2, \cdots, v_m^n$ in G with a partition of $V_1, v_2, \cdots, V_n$ of $V(G)$ such that v_m^i is important than v_1^i, v_1^i is important than $v_2^i \cdots$, and v_j^i is important than $v_{j+1}^i, \cdots$, important than u_m^i for $\forall 1 \le i \le n$, $j \geqslant 1$, we call $v_m^i, u_m^i, 1 \le i \le n$ mother vertices and father vertices. Particularly, if $n = 1$ and there are no father vertices in a graph G, we call such a graph G 1-mother graph, seeing Figure 1.*

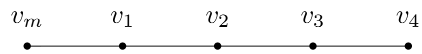
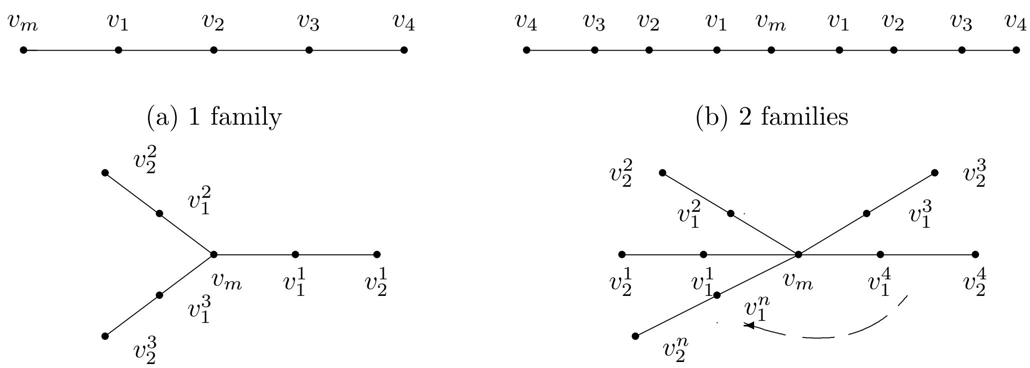


Figure 1

Now we will classify the 1-mother vertex graph with respect to the number of the family which contacts with the mother vertex as follows:

Definition 3.2 *A 1-mother vertex graph with n families of vertices is a graph G_m which it's vertex-set has the form $V\left\{v_m; v_1^1, v_2^1, v_3^1, \cdots; v_1^2, v_2^2, v_3^2, \cdots; \cdots; v_1^n, v_2^n, v_3^n, \cdots\right\}$, where $v_1^i, v_2^i, v_3^i, \cdots$ is the i-th family, seeing Figure 2.*



(c) 3 families

(d) n families

Figure 2

Definition 3.3 *Any edge has v_m as a vertex is called a mother edge.*

In Figure 2, there is a one mother edge in (a), two mother edges in (b), three mother edges in (c) and n mother edges in (d).

Note 1) The families of vertices in a 1-mother vertex graph with n families not necessary have the same number of vertices , seeing Figure 3.

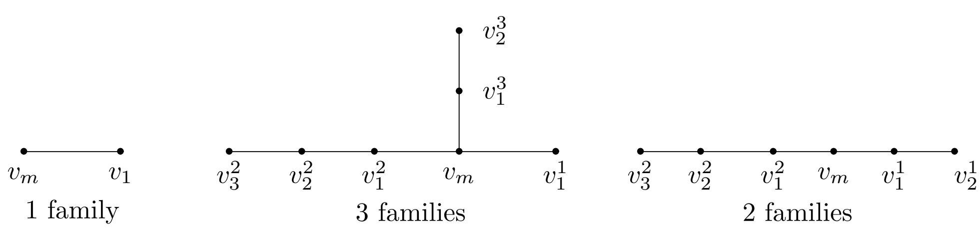


Figure 3

2) The following graph is not 1-mother vertex graph, seeing Figure 4.

Figure 4

Definition 3.4 *An empty 1-mother vertex graph is an 1-mother vertex graph with no vertices and no edges.*

Definition 3.5 *A simple 1-mother vertex graph is an 1-mother vertex graph with no loops and no multiple edges, seeing Figure 3.*

Definition 3.6 *A multiple 1-mother vertex graph is an 1-mother vertex graph allows multiple edges and loops, seeing Figure 5.*

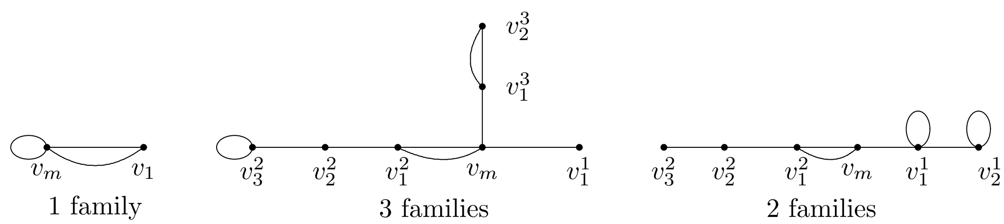
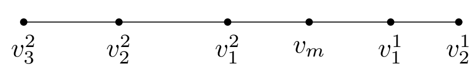
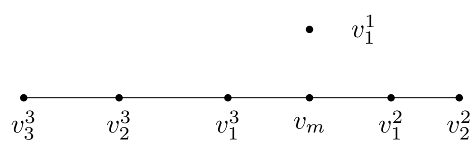


Figure 5

Definition 3.7 *A connected 1-mother vertex graph is 1-mother vertex graph that in one piece and the one which splits into several pieces is disconnected,seeing Figure 6.*



(a) A connected1-mother vertex graph with 2 families



(a) A connected1-mother vertex graph with 2 families

Figure 6

Note The following graph is not disconnected 1-mother vertex graph and also is not 1-mother vertex graph.

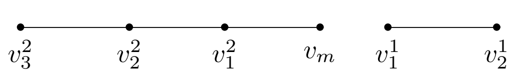


Figure 7

Definition 3.8 *A graph H_m^i is said to be main supgraph of G_m^n ,where $n, i \in z^+$ and $i \le n$, if $V(H_m^i) \subseteq V(G_m^n)$, $E(H_m^i) \subseteq E(G_m^n)$ and $v_m \in V(H_m^i)$.*

Proposition 3.1 *The main supgraph H_m^i of G_m^n is 1-mother vertex graph.*

Definition 3.9 *A graph H is a supgraph of G_m^n if $V(H) \subseteq V(G_m^n)$, $E(H) \subseteq E(G_m^n)$ and $v_m \notin V(H_m^i)$.*

Proposition 3.2 *A supgraph H of G_m^n is not 1-mother vertex graph.*

Example 3.1 As shown in Figure 8, H_m^2 is a main supgraph of G_m^2 and H is a supgraph of G_m^2.

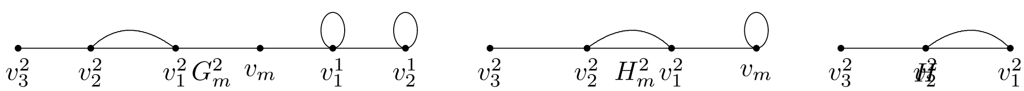


Figure 8

Definition 3.10 *An oriented 1-mother vertex graph is a pair (V, E) where V is finite non empty set of vertices and E is a set of ordered pairs of distinct elements of E with the property that if $(v, w) \in E$, then $(w, v) \notin E$, where the element (v, w) denote the edge from v to w, seeing Figure 9.*

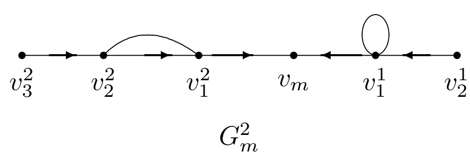


Figure 9

Definition 3.11 *Let G_m^n be a 1-mother vertex graph, with n-families of vertices. The adjacency matrix A(G_m^n) is the (n+1)x(n+1) matrix in which the entry in row i and column j is matrix its elements are the number of edges joining the families i and j.*

Definition 3.12 *Let G_m^n be a 1-mother vertex graph, with n-families. The incidence matrix I(G_m^n) is the (n+1) x n matrix in which the entry in row i and column j is matrix its elements is l if vertex in family i incident with edge in family j and 0 otherwise.*

Example 3.2 The adjacency matrix and the incidence matrix of a 1-mother vertex graph G_m^2

as shown in Figure 9 are given by

$$A(G_m^2)=\begin{bmatrix}0&1&0&1&0&0\\1&1&1&0&0&0\\0&1&0&0&0&0\\1&0&0&0&2&0\\0&0&0&2&0&1\\0&0&0&0&1&0\end{bmatrix}\qquad I(G_m^2)=\begin{bmatrix}1&1^1&0&0&0&0\\0&1&1&0&0&0\\1&0&0&1&0&0\\0&0&0&1&1&0\\0&0&0&1&1&0\\0&0&0&0&1&1\end{bmatrix}$$

where the symbol 1^1in the matrix in the row 1 and column 2 of the incidence matrix means that there exists a loop at the vertex $v_1^1$with the edge e_1^1.

Theorem *A* *A complete 1-mother vertex graph is not defined.*

Proof Let there exist a complete 1-mother vertex graph. Then this mean that every two distinct vertices are joined which is contradict with the definition of the 1-mother vertex graph. Hence the complete 1-mother vertex graph is not define. □

New we will define the union of any 1-mother vertex graphs as follows:

Definition 3.13 *The union of G_m^s and G_m^v , denoted $G_m^s \cup G_m^v$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.*

Proposition 3.3 *The union of any 1-mother vertex graphs is 1-mother vertex graph if* $v_m \in V_1 \cap V_2$.

Proof Let we have two 1-mother vertex graphs, the union of these graphs has one of two types.

1) If $v_m \in \mathrm{V}_1 \cap V_2$, i.e. the new graph has one mother vertex, then the new graph is 1-mother vertex graph, seeing Figure 10.a.

2) If $v_m \notin \mathrm{V}_1 \cap V_2$, i.e. the new graph has more than one mother vertex, then the new graph is not 1-mother vertex graph, seeing Figure 10-b. □

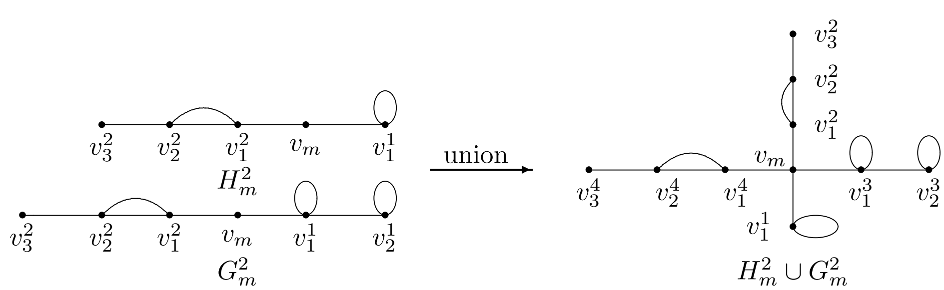


Figure 10-a

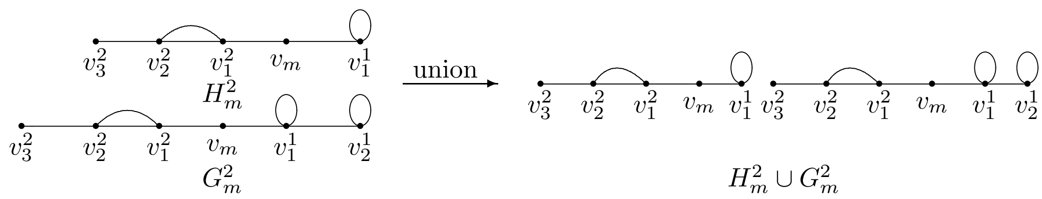


Figure 10-b

The intersection of 1-mother vertex graphs will be defined as follows:

Definition 3.14 *The intersection of G_m^s and G_m^v, denoted $G_m^s \cap G_m^v$ is the graph with vertex set $V_1 \cap V_2$ and edge set $E_1 \cap E_2$.*

Proposition 3.4 *The intersection of any number of 1-mother vertex graphs is 1-mother vertex graph if $v_m \in V_1 \cap V_2$ or $V_1 \cap V_2 = \phi$ and $E_1 \cap E_2$.*

Proof Let we have n number of 1-mother vertex graphs, the intersection of these graphs has one of two types.

1) If $v_m \in V_1 \cap V_2 \cdots \cap V_n$, i.e. the new graph has one mother vertex, then the new graph is 1-mother vertex graph, seeing Figure 11-a.

2) If $v_m \notin V_1 \cap V_2 \cdots \cap V_n = \phi$, i.e. the new graph is the empty 1-mother vertex graph.

3) If $v_m \notin V_1 \cap V_2 \cdots \cap V_n \neq \phi$, i.e. the new graph has more than one mother vertex, then the new graph is not 1-mother vertex graph, see Figure 11-b. □

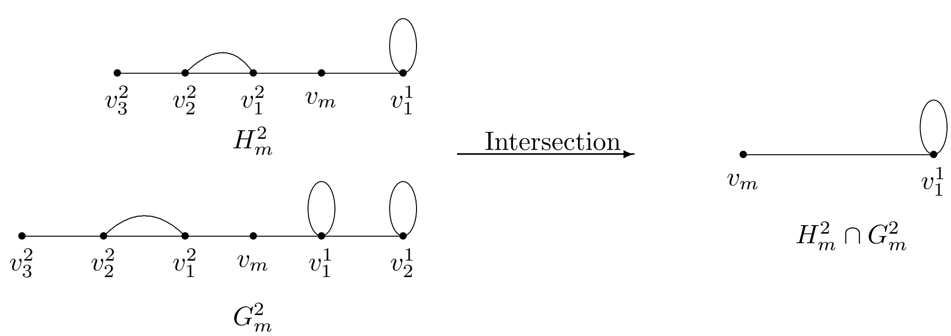


Figure 11-a

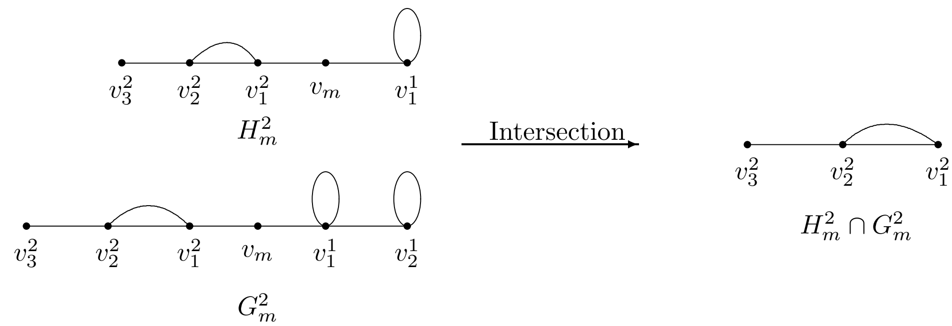
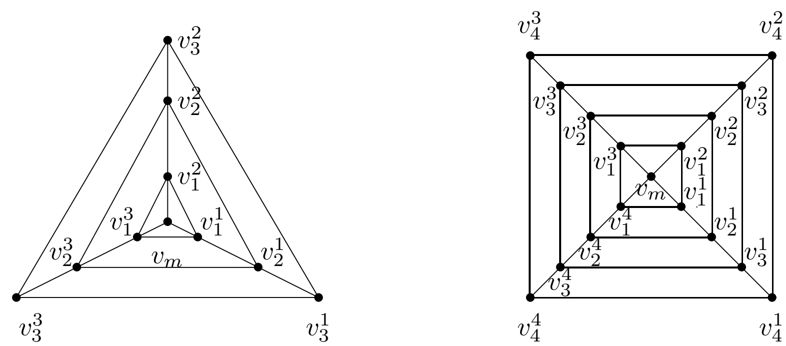


Figure 11-b

In this section we will define special types of 1-mother vertex graphs.

Definition 3.15 *A spider mother graph S_m^n is 1-mother vertex graph has the form as shown in Figure 12.*



(a) Spider mother graph with 3 families (b) Spider mother graph with 4 families

Figure 12

Note The least number of families which the spider graph has is three.

Definition 3.16 *A tree mother graph T_m^n is 1-mother vertex graph has the form as shown in Figure 13.*

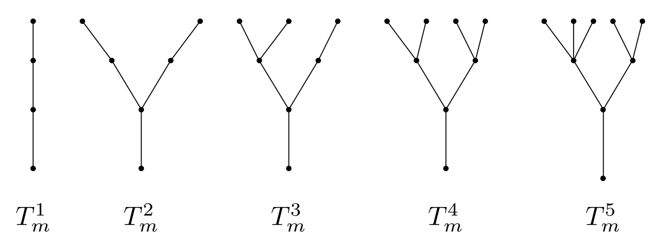


Figure 13

Example 3.3 The adjacency matrix of the spider graph as shown in Figure 12-a are given by

$$\begin{bmatrix} 0&1&0&0&1&0&0&1&0&0\\ 1&0&1&0&1&0&0&1&0&0\\ 0&1&0&1&0&1&0&0&1&0\\ 0&0&1&0&0&0&1&0&0&1\\ 1&1&0&0&0&1&0&1&0&0\\ 0&0&1&0&1&0&1&0&1&0\\ 0&0&0&1&0&1&0&0&0&1\\ 1&1&0&0&1&0&0&0&1&0\\ 0&0&1&0&0&1&0&1&0&1\\ 0&0&0&1&0&0&1&0&1&0 \end{bmatrix}$$

the existence of the unit matrix in column I and raw j means that the family in the column I and the family in the raw j have the relation between the vertices which have the same order.

Definition 3.17 *An orbit mother graph is 1-mother vertex graph is a 1-mother vertex graph containing no edges and the elements in the same family have the same distance from the mother vertex, seeing Figure 14.*

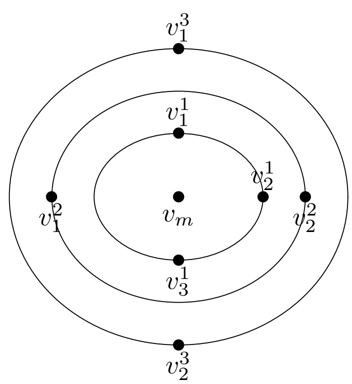


Figure 14

§4. **Applications**

(1) The solar system is orbit mother graph.

(2) If we illustrate the nervous system by using the graph we find that the nervous system is 1-mother vertex graph, seeing Figure 16.

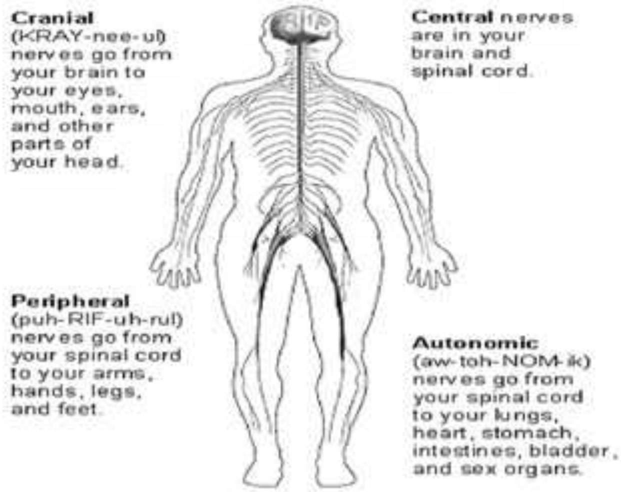


Figure 16

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Perfect understanding will sometimes almost extinguish pleasure.

By A.E.Housman, a British scholar and poet

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[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009. [12]W.S. Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.*, Vol.1, 1-19(2007).
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